Good Jump and Bad Jump Risk Matters: Evidence from S&P500 Returns and Options

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Abstract

The understanding of the relationship between an asset's expected return and its volatility is pivotal in asset pricing. In this paper, we extend the asymmetric double exponential jump-diffusion model grounded in the affine generalized autoregressive conditional heteroskedastic (GARCH) framework. We propose a model within the affine GARCH setting that uses two exponential distributions to separately model good and bad jumps. Furthermore, we deduce a closed-form solution for option pricing within this model structure. Our results suggest that the integration of jump components into the variance process significantly bolsters model estimation performance—the bad jump component markedly outstrips its good counterpart in contribution. In our empirical evaluation, we discern the variance risk premiums attributable to these good and bad jumps through model estimation. A cross-sectional regression reveals that both variance risk premiums serve as priced risk factors. Moreover, a time-series examination underscores the prevailing role of the bad jump variance risk premium in forecasting returns.

Keywords: Variance risk premium; Good jump and bad jump; Option pricing; Cross-sectional regression; Time series analysis

JEL Codes: C51, C58, G12, G13

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1 Introduction

Understanding the interplay between financial market risk measures and expected market returns is a critical endeavor in asset pricing. The literature abounds with evidence of time-variant volatility and volatility clustering within market returns, phenomena that introduce uncertainty into price risk and, consequently, give rise to risk premiums. A recognized approach to effectively capture these dynamics is to include jump components in market return models (Bates (1991); Christoffersen, Jacobs, and Ornthanalai (2012); Christoffersen, Feunou, and Jeon (2015); Yang (2018); Chang et al. (2019)). Chang et al. (2019) identify a divergence between realized volatility that incorporates a jump component and its risk-neutral equivalent, coining this discrepancy the jump variance risk premium (JVRP). Notably, this premium has been observed to be both negative and variable over time. In financial markets, good jumps typically stem from investor optimism about overall economic prospects, while bad jumps are usually the result of market trepidation (Kilic and Shaliastovich (2019)). However, individual investor sentiments regarding market conditions are not homogenous. Consequently, an increasing body of research has highlighted the significance of differentiating between upward and downward market volatilities (Feunou, Jahan-Parvar, and Tédongap (2013); Bekaert, Engstrom, and Ermolov (2015); Kilic and Shaliastovich (2019)). Thus, distinguishing between good and bad jumps is anticipated to be crucial within the JVRP framework.

In this study, we demonstrate the importance of both good and bad jump innovations. We use an option pricing framework to examine the roles played by these jumps in the variance risk premium (VRP). Therefore, option valuation requires a robust pricing formula. From asset returns and variance dynamic processes to the kernel's pricing design, each phase informs the option valuation paradigm. Additionally, our estimations provide insights into the VRP's associations with good and bad jumps, aiding our exploration of their respective pricing and return predictability on a cross-sectional basis. Concerning model selection, we first analyze a real-time return series. The top panel of Figure 1 displays the time series of S&P 500 index returns from 1996 to 2020. Without loss of generality, we define returns exceeding the mean return plus three times the standard deviation as good jumps (middle panel of Figure 1) and returns falling below the mean return minus three times the standard deviation as bad jumps (bottom panel of Figure 1). Observing the middle and bottom panels of Figure 1, we note that when a good (bad) jump occurs, the likelihood of a subsequent good (bad) jump increases, indicating that both good jump volatility and bad jump volatility exhibit time-varying and clustering properties; hence, incorporating both good and bad jumps in models is critical. Consequently, our approach aligns with the generalized autoregressive conditional heteroskedastic (GARCH) option pricing framework proposed by Heston and Nandi (2000). The GARCH framework accurately characterizes both time-varying and clustering dy-

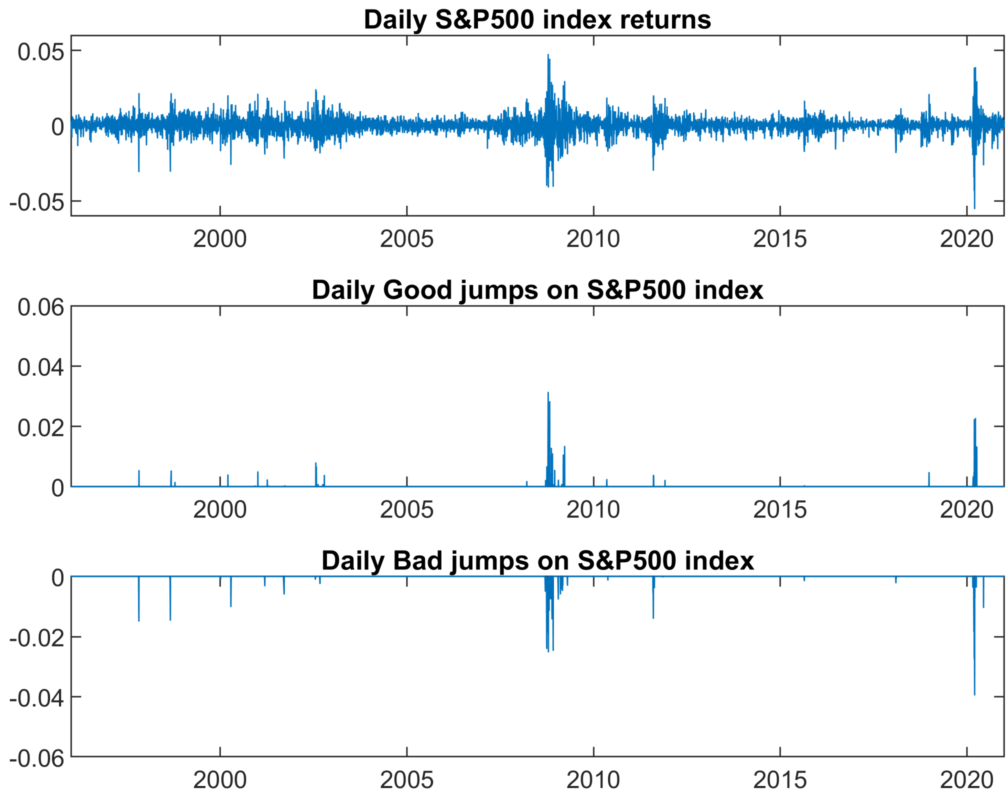

Figure 1: **Daily return, good jumps, and bad jumps on the S&P 500 index**
Note: The sample period is from January 1996 to December 2020. The upper figure presents daily S&P500 index returns. Without loss of generality, in the middle figure, a good jump is defined as a return that exceeds the average return plus thrice the standard deviation, whereas in the bottom figure, a bad jump is defined as a return less than the mean return minus three times the standard deviation.

namics. Furthermore, compared to stochastic volatility model, the GARCH framework facilitates model estimation and interpretation (Christoffersen, Jacobs, and Ornthanalai (2012)). Therefore, a plethora of option valuation literature conducts in-depth studies within the context of the GARCH model (Kanniainen, Lin, and Yang (2014); Oh and Park (2023)).

In the proposed modeling framework, the asset return process builds upon the structure delineated by Yang (2018), incorporating innovations that manifest as normal, good jumps, and bad jumps. Yang (2018) amalgamates both normal and jump innovations, with the jump innovations depicted using the asymmetric exponential jump-diffusion framework known as the Kou model. This model proficiently addresses the limitations of presuming jump innovations to align with a normal distribution (Kou (2002); Kou and Wang (2004)). To account for varied investor sentiments toward asset returns, we segment the Kou model into two distinct exponential distributions, individually capturing good and bad jump innovations. This refined structure not only accommodates the spectrum of investor sentiments but also elucidates the distinct roles of good and bad jumps within the risk premium spectrum. In the variance dynamic, alongside the heterogeneity of the standard component, we delineate the intensities associated with both good and bad jumps. In contrast to Ornthanalai (2014), who omits the influence of jumps on the heterogeneity of the normal component and jump intensities, ample literature emphasizes the significance of embedding jump dynamics within volatility for precise option valuation (Eraker (2004); Christoffersen, Jacobs, and Ornthanalai (2012); Yang (2018); Chang et al. (2019)). Hence, our model is designed to simultaneously capture the influences of both types of jumps on the normal component's heterogeneity and their specific intensities. Our pricing kernel integrates four state variables: market returns, good jumps, bad jumps, and continuous variance dynamics. Adhering to Christoffersen, Heston, and Jacobs (2013), when continuous variance dynamics are embedded, our kernel design intriguingly mirrors a non-linear U-shaped projection on market returns. Moreover, our model facilitates an asymmetrical impact of the risk premiums from good and bad jumps on the pricing kernel, ensuring a robust bridge in measure transformation during option valuation. Drawing from the insights of Heston and Nandi (2000), we derive a closed-form option pricing solution within a model structure that entails normal, good, and bad jump innovations.

We estimate our model's parameters by employing a joint maximum likelihood estimation (joint MLE) method coupled with data from the S&P 500 Index and its options spanning 1996 to 2020. Joint MLE, where index returns and options can be jointly considered, has the advantage of being resilient against overfitting. We aspire to determine the specific repercussions of good and bad jumps within the variance process. Four model scenarios are considered: one devoid of any jumps, one with only good jumps, one with only bad jumps, and one with both good and bad jumps. Observations suggest that the model with no jumps is the least effective, consistent with results in previous studies. Interestingly, the model with only bad jumps outperforms that with only good jumps. This reinforces the general perception that decreases in asset returns during bad jumps are greater in magnitude than increases in asset returns during good jumps. It underscores the potency of including bad jumps to preemptively gauge impending variance shifts. Furthermore, our findings reveal that risk premiums stemming from normal and bad jump innovations are positive, denoting appropriate compensatory premiums for investors bearing these risks. Conversely, the risk premium linked to good jump innovations is negative, suggesting that investors might bear a marginal cost to capitalize on the upside potential of asset returns; this is consistent with the logic of investors shouldering costs for hedging tools.

In our empirical analysis, we commence by employing cross-sectional regression to ascertain the pricing of the continuous variance risk premium (CVRP), good jump variance risk premium (GJVRP), and bad jump variance risk premium (BJVRP) in relation to expected stock returns. In doing so, we utilize not only the Fama-MacBeth regression approach but also the three-pass regression technique, as introduced by Giglio and Xiu (2021), to evaluate the risk premiums attributable to each of these factors. Considering the susceptibility of regression models to overlook critical explanatory variables, such oversights can lead to biases in estimating risk premiums. To address this, Giglio and Xiu (2021) improve upon the traditional two-stage regression approach, which resembles the Fama-MacBeth methodology, by devising a three-pass regression specifically tailored to reduce estimation errors caused by omitted variables. Our preliminary results indicate that the risk premiums for all types of VRP under investigation are statistically significant, suggesting that these VRPs serve as priceable risk factors. The risk premium for the BJVRP emerges as the most substantial, displaying a magnitude comparable to that of the aggregate JVRP. This finding further elucidates the cross-sectional results documented by Chang et al. (2019), which highlighted that the risk premium associated with the JVRP is primarily driven by the BJVRP. This evidence accentuates the BJVRP's effectiveness in reflecting investor anticipation of unfavorable market conditions and suggests that such prevalent bearish sentiments contribute to the heightened expected stock returns that are associated with the BJVRP.

Subsequently, we explore the potential of the CVRP, GJVRP, and BJVRP to forecast market returns. While numerous academic contributions have validated the efficacy of the VRP in prognosticating subsequent market returns (Bakshi and Kapadia (2003); Bollerslev, Tauchen, and Zhou (2009); Byun et al. (2015); Li and Zinna (2018)), Chang et al. (2019) have broadened this narrative by revealing that the predictive power of the VRP is overwhelmingly influenced by the JVRP. Their investigations predominantly centered around forecasting market yields spanning horizons of 6 months up to a year. Seeking more comprehensive insights, we expand the prediction window, ranging from a concise one-month interval to an extensive 24-month period. Consistent with extant literature, our preliminary data indicates that the VRP presents a discernibly negative coefficient when forecasting market yields exceeding a six-month horizon, primarily attributable to the JVRP's influence. Concurrently, the CVRP's impact on predicting market returns emerges as statistically insignificant, resonating with Chang et al. (2019)'s conclusions. A deeper analysis unveils that the JVRP's significance is largely propelled by the BJVRP, with a diminished BJVRP correlating with augmented prospective market yields. Conversely, the GJVRP's prowess in forecasting market yields appears to be inconsequential.

Overall, our study makes a significant contribution to the literature on option pricing and the empirical asset pricing involving VRPs. We employ a GARCH option pricing model, considering both good jump and bad jump innovations within asset returns. Additionally, our model permits both good and bad jump innovations to simultaneously influence the subsequent period's variance. This methodological innovation ameliorates limitations found in previous literature. Firstly, traditional jump innovations assume that jump size follows a normal distribution, rendering them incapable of adequately capturing the asymmetry between good and bad jumps (Kou, Yu, and Zhong (2017)). Secondly, while Yang (2018) adeptly applies the asymmetric exponential jump-diffusion model to GARCH option pricing, our contribution extends Yang (2018)'s model. We further decompose the jump component into good jump and bad jump innovations. This presupposes that different investors have varying perspectives on asset returns, and it also facilitates our subsequent examination of the empirical performance of GJVRP versus BJVRP. In conclusion, across both cross-sectional and time-series analyses, our study further illuminates that the BJVRP is the pivotal risk factor driving the overall JVRP.

The remainder of this paper is structured as follows: Section 2 discusses GARCH asset dynamics incorporating normal, good jump, and bad jump innovations and elaborates on the derivation of the risk-neutral process and the closed-form option pricing formula. Section 3 describes the estimation methods and provides a comparative analysis of model performance. Section 4 examines the predictive power of the estimated GJVRPs and BJVRPs for stock market returns, both in cross-sectional and time-series contexts. Section 5 concludes the paper.

2 GARCH model with good and bad jump dynamics

2.1 The asset return process and variance dynamic process

To consider the effect of good and bad jumps on asset returns, we extend the general return process by splitting the jump component into two exponential distributions, representing good and bad jump innovations, respectively. The asset return process under the physical measure is specified as follows

$$\begin{aligned} R_t &\equiv \log\left(\frac{S_t}{S_{t-1}}\right) \\ &= r_t + \left(\lambda_z - \xi_z(1)\right) h_{z,t} + \left(\lambda_g - \xi_g(1)\right) h_{g,t} + \left(\lambda_b - \xi_b(-1)\right) h_{b,t} + z_t + y_{g,t} - y_{b,t}, \end{aligned} \tag{1}$$

where S_t is the asset price at time t, R_t is the log-return, and r_t is the risk-free rate. The normal component z_t follows a normal distribution with heterogeneous variances and is defined as

$$z_t \sim N\left(0, h_{z,t}\right).$$

The good jump innovation $y_{g,t}$ and bad jump innovation $y_{b,t}$ follow a compound Poisson process can be written as

$$y_{g,t} = \sum_{j=0}^{n_{g,t}} x_{g,t}^j \sim CPJ\left(h_{g,t}, \mu_g\right),$$

$$y_{b,t} = \sum_{j=0}^{n_{b,t}} x_{b,t}^j \sim CPJ\left(h_{b,t}, \mu_b\right),$$

where $x_{g,t}^j$ and $x_{b,t}^j$ follow exponential distributions with jump size μ_g and μ_b, respectively, whereas $n_{g,t}$ and $n_{b,t}$ denote Poisson distributions with jump intensity $h_{g,t}$ and $h_{b,t}$, respectively. λ_z, λ_g, and λ_t are the market prices of risks of the normal, good jump, and bad jump innovations, respectively. $\xi_z(t) = t^2/2$, $\xi_g(t) = \mu_g/(\mu_g - t) - 1$ and $\xi_b(t) = \mu_b/(\mu_b - t) - 1$. Finally, $\xi_z(1)h_{z,t}$, $\xi_g(1)h_{g,t}$, and $\xi_b(-1)h_{b,t}$ denote the convexity adjusted terms. The conditional expectation of asset price is

$$E_{t-1}^{\mathbb{P}}\left[\frac{S_t}{S_{t-1}}\right] = e^{r_t + \lambda_z h_{z,t} + \lambda_g h_{g,t} + \lambda_b h_{b,t}}. \tag{2}$$

We define $\lambda_z h_{z,t} + \lambda_g h_{g,t} + \lambda_b h_{b,t}$ as the conditional equity premium. For investors, risks arising from the normal component and bad jump innovations are undesirable, leading to their respective risk premiums being positive, that is, $\lambda_z > 0$ and $\lambda_b > 0$. By contrast, risks stemming from good jump innovation are favored by investors because these jumps have a positive impact on returns. As a result, investors are willing to incur some costs to hold onto assets with good jump innovation, akin to holding hedging instruments. Therefore, the risk premium for good jump innovation is negative despite $\lambda_g < 0$.

The dynamics of variance for the normal component, good jump intensity, and bad jump intensity are an extended affine GARCH dynamic with good and bad jump innovations given by

$$h_{z,t} = w_z + b_z h_{z,t-1} + \frac{a_z}{h_{z,t-1}}\left(z_{t-1} - c_z h_{z,t-1}\right)^2 + d_z y_{g,t-1} + e_z y_{b,t-1}, \tag{3}$$

$$h_{g,t} = w_g + b_g h_{g,t-1} + \frac{a_g}{h_{z,t-1}}\left(z_{t-1} - c_g h_{z,t-1}\right)^2 + d_g y_{g,t-1} + e_g y_{b,t-1}, \tag{4}$$

$$h_{b,t} = w_b + b_b h_{b,t-1} + \frac{a_b}{h_{z,t-1}}\left(z_{t-1} - c_b h_{z,t-1}\right)^2 + d_b y_{g,t-1} + e_b y_{b,t-1}, \tag{5}$$

with initial conditions $h_{z,0}$, $h_{g,0}$, and $h_{b,0}$. This formulation aptly encapsulates the asymmetry and clustering characteristics of normal variance, good jump intensity, and bad jump intensity. Moreover, it permits the influences of both good and bad jump innovations on normal variance, good jump intensity, and bad jump intensity to be concurrently considered.

2.2 Conditional higher moments of return process

Through the moment generating functions of the normal distribution and the compound Poisson process, we can derive the conditional higher-order moments under Equation (1):

$$Var_{t-1}(R_t) = h_{z,t} + \frac{2}{\mu_g^2}h_{g,t} + \frac{2}{\mu_b^2}h_{b,t}, \tag{6}$$

$$Skew_{t-1}(R_t) = \frac{6/\mu_g^3 h_{g,t} - 6/\mu_b^3 h_{b,t}}{(h_{z,t} + 2/\mu_g^2 h_{g,t} + 2/\mu_b^2 h_{b,t})^{3/2}}, \tag{7}$$

$$Kurt_{t-1}(R_t) = \frac{24/\mu_g^4 h_{g,t} + 24/\mu_b^4 h_{b,t}}{(h_{z,t} + 2/\mu_g^2 h_{g,t} + 2/\mu_b^2 h_{b,t})^{2}}, \tag{8}$$

where $Skew_{t-1}(R_t)$ denotes the conditional skewness of the return process and $Kurt_{t-1}(R_t)$ represents the conditional kurtosis of the return process. The direction of the conditional skewness depends on the difference between the good jump intensity paired with the good jump size and the bad jump intensity paired with the bad jump size. Through our model, we can obtain time-varying conditional skewness and conditional kurtosis. Harvey and Siddique (2000) have already substantiated that time-varying conditional skewness is a crucial concept in the asset pricing domain.

2.3 The pricing kernel

Because the projection of the pricing kernel on stock returns is nonmonotonic (Christoffersen, Heston, and Jacobs (2013)), we assume a variance-dependent pricing kernel with good jumps and bad jumps innovations given by

$$\log\left(\frac{M_t}{M_{t-1}}\right) = \delta_t - \gamma R_t + \gamma_G y_{g,t} - \gamma_B y_{b,t} + \gamma_{VZ}(h_{z,t+1} - h_{z,t}), \tag{9}$$

where R_t represents the logarithmic asset return and $y_{g,t}$ and $y_{b,t}$ indicate the good jump innovation and the bad jump innovation, respectively. $h_{z,t+1} - h_{z,t}$ is the variance premium of the stochastic continuous component. The adjustment coefficient δ_t is used to ensure that $E^{\mathbb{P}}_{t-1}[M_t/M_{t-1}]$ is equal to the risk-free interest rate.

This pricing kernel not only accounts for the VRP of asset returns and the stochastic continuous component but also considers the impact of good jumps and bad jumps on pricing. For investors, the inherent risk associated with holding assets demands a corresponding risk compensation. Thus, both the equity premium and the bad jump premium are positive. Given that the marginal utility is a diminishing function of asset returns, the signs preceding the equity premium, denoted as γ, and the bad jump premium, denoted as γ_B, are negative $(-)$. By contrast, because good jumps always have a positive relationship with returns, investors are inclined to incur additional costs to leverage the benefits of such positive jumps; this phenomenon is similar to hedging. Consequently, both the good jump premium and variance premium are negative. Therefore, the signs preceding the good jump premium γ_G and the variance premium γ_{VZ} are positive $(+)$. This pricing kernel is more general than those in the literature. When $\gamma_G = \gamma_B = \gamma_{VZ} = 0$, the pricing kernel degenerates into a power utility pricing kernel. When $\gamma_G = \gamma_B = 0$, the pricing kernel degenerates into the pricing kernel designed by Christoffersen, Heston, and Jacobs (2013).

2.4 The risk-neutral measure and option valuation

With the pricing kernel in Equation (9), we can obtain the corresponding Radon–Nikodým derivative function as

$$\frac{d\mathbb{Q}_t/d\mathbb{P}_t}{d\mathbb{Q}_{t-1}/d\mathbb{P}_{t-1}} = \frac{\exp\left(-\Lambda_Z z_t + \Lambda_{VZ}(z_t^2/h_{z,t}) - \Lambda_G y_{g,t} - \Lambda_B y_{b,t}\right)}{E_{t-1}^{\mathbb{P}}\left[\exp\left(-\Lambda_Z z_t + \Lambda_{VZ}(z_t^2/h_{z,t}) - \Lambda_G y_{g,t} - \Lambda_B y_{b,t}\right)\right]}, \tag{10}$$

where $\Lambda_Z = \gamma + 2a_z c_z \gamma_{VZ}$, $\Lambda_{VZ} = a_z\gamma_{VZ}$, $\Lambda_G = \gamma - \gamma_G - d_z\gamma_{VZ}$, and $\Lambda_B = -\gamma + \gamma_B - e_z\gamma_{VZ}$ are equivalent martingale measure coefficients.

Lemma 1 *We can derive the closed-form of the Radon-Nikodým derivative where*

$$\begin{aligned} &E_{t-1}^{\mathbb{P}}\left[\exp\left(-\Lambda_Z z_t + \Lambda_{VZ}(z_t^2/h_{z,t}) - \Lambda_G y_{g,t} - \Lambda_B y_{b,t}\right)\right] \\ &\quad = \exp\left(-\frac{\log(\phi)}{2} + \frac{\Lambda_Z^2}{2\phi}h_{z,t} + \left(\frac{\mu_g}{\mu_g + \Lambda_G} - 1\right)h_{g,t} + \left(\frac{\mu_b}{\mu_b + \Lambda_B} - 1\right)h_{b,t}\right), \end{aligned} \tag{11}$$

where $\phi = 1 - 2\Lambda_{VZ}$.
Proof. *See Appendix A.* ■

Proposition 1 *If the asset return process under the physical probability measure is represented by Equation (1), then the risk-neutral probability measure ($\mathbb{Q}$), as characterized by the Radon–Nikodým derivative in Equation (10), constitutes an equivalent martingale measure if, and only if*

$$\lambda_z - \frac{1}{2} + \frac{1 - 2\Lambda_Z}{2(1 - 2\Lambda_Z)} = 0, \tag{12}$$

$$\lambda_g - \left(\frac{\mu_g}{\mu_g - 1} - 1\right) - \left(\frac{\mu_g}{\mu_g + \Lambda_G} - 1\right) + \left(\frac{\mu_g}{\mu_g + (\Lambda_G - 1)} - 1\right) = 0, \tag{13}$$

and

$$\lambda_b - \left(\frac{\mu_b}{\mu_b+1} - 1\right) - \left(\frac{\mu_b}{\mu_b+\Lambda_B} - 1\right) + \left(\frac{\mu_b}{\mu_b+(\Lambda_B+1)} - 1\right) = 0. \tag{14}$$

Proof. *See Appendix B.* ■

Given the Radon–Nikodým derivative in Equation (10) and Proposition 1, we then derive the asset return process and variance process under the $\mathbb{Q}$-measure.

Proposition 2 *The asset return dynamic in Equation (1) and variance dynamics in Equation (3) - (5) under* $\mathbb{Q}$*-measure can be written as*

$$R_t = \log\left(\frac{S_t}{S_{t-1}}\right) = r_t - \xi_z^*(1)h_{z,t}^* - \xi_g^*(1)h_{g,t}^* - \xi_b^*(-1)h_{b,t}^* + z_t^* + y_{g,t}^* - y_{b,t}^*, \tag{15}$$

$$h_{z,t}^* = w_z^* + b_z h_{z,t-1}^* + \frac{a_z^*}{h_{z,t-1}^*}\left(z_{t-1}^* - c_z^* h_{z,t-1}^*\right)^2 + d_z^* y_{g,t-1}^* + e_z^* y_{b,t-1}^*, \tag{16}$$

$$h_{g,t}^* = w_g^* + b_g h_{g,t-1}^* + \frac{a_g^*}{h_{z,t-1}^*}\left(z_{t-1}^* - c_g^* h_{z,t-1}^*\right)^2 + d_g^* y_{g,t-1}^* + e_g^* y_{b,t-1}^*, \tag{17}$$

$$h_{b,t}^* = w_b^* + b_b h_{b,t-1}^* + \frac{a_b^*}{h_{z,t-1}^*}\left(z_{t-1}^* - c_b^* h_{z,t-1}^*\right)^2 + d_b^* y_{g,t-1}^* + e_b^* y_{b,t-1}^*, \tag{18}$$

where $z_t^* \sim N(0, h_{z,t}^*)$*, is the risk-neutral normal innovation* $y_{g,t}^* \sim CPJ\left(h_{g,t}^*, \mu_g^*\right)$ *is the risk-neutral good jump innovation,* $y_{b,t}^* \sim CPJ\left(h_{b,t}^*, \mu_b^*\right)$ *is the risk-neutral bad jump innovation,* $h_{z,t}^* = h_{z,t}/\phi$, $h_{g,t}^* = h_{g,t}\Pi_g$, $h_{b,t}^* = h_{b,t}\Pi_b$, $\mu_g^* = \mu_g + \Lambda_G$, $\mu_b^* = \mu_b + \Lambda_B$, $\xi_z^*(t) = t^2/2$, $\xi_g^*(t) = \mu_g^*/(\mu_g^* - t) - 1$, $\xi_b^*(t) = \mu_b^*/(\mu_b^* - t) - 1$, $w_z^* = w_z/\phi$, $a_z^* = a_z/\phi^2$, $c_z^* = c_z\phi + \Lambda_Z$, $d_z^* = d_z/\phi$, $e_z^* = e_z/\phi$, $w_g^* = w_g\Pi_g$, $a_g^* = a_g\Pi_g/\phi$, $c_g^* = c_g\phi + \Lambda_Z$, $d_g^* = d_g\Pi_g$, $e_g^* = e_g\Pi_g$, $w_b^* = w_b\Pi_b$, $a_b^* = a_b\Pi_b/\phi$, $c_b^* = c_b\phi + \Lambda_Z$, $d_b^* = d_b\Pi_b$, $e_b^* = e_b\Pi_b$, $\Pi_g = \mu_g/\mu_g^*$, *and* $\Pi_b = \mu_b/\mu_b^*$.

Proof. *See Appendix C.* ■

To derive an option pricing model, we deduce the moment generating function of the asset return dynamics under the $\mathbb{Q}$-measure based on the results from Proposition 2.

Proposition 3 *First, we consider the moment generating function of the multiperiod aggregate return as follows:*

$$\begin{aligned} f^{\mathbb{Q}}(\psi; t, T) &\equiv E_t^{\mathbb{Q}}\left[\exp\left(\psi \sum_{j=1}^{T-t} R_{t+j}\right)\right] \\ &= \exp\left(A(\psi;t,T) + B(\psi;t,T)h_{z,t+1}^* + C(\psi;t,T)h_{g,t+1}^* + D(\psi;t,T)h_{b,t+1}^*\right), \end{aligned} \tag{19}$$

where T denotes the terminal date,

$$
\begin{aligned}
A(\psi;t,T) = {} & \psi r_{t+1} + A(\psi;t+1,T) + B(\psi;t+1,T)w_z^* + C(\psi;t+1,T)w_g^* + D(\psi;t+1,T)w_b^* \\
& - \frac{1}{2}\log\left(1 - 2B(\psi;t+1,T)a_z^* - 2C(\psi;t+1,T)a_g^* - 2D(\psi;t+1,T)a_b^*\right),
\end{aligned}
$$

$$
\begin{aligned}
B(\psi;t,T) = {} & -\frac{\psi}{2} + B(\psi;t+1,T)(b_z + a_z^* c_z^{*2}) + C(\psi;t+1,T)a_g^* c_g^{*2} + +D(\psi;t+1,T)a_b^* c_b^{*2} \\
& + \frac{(\psi - 2B(\psi;t+1,T)a_z^* c_z^* - 2C(\psi;t+1,T)a_g^* c_g^* - 2D(\psi;t+1,T)a_b^* c_b^*)^2}{2(1 - 2B(\psi;t+1,T)a_z^* - 2C(\psi;t+1,T)a_g^* - 2D(\psi;t+1,T)a_b^*)},
\end{aligned}
$$

$$
\begin{aligned}
C(\psi;t,T) = {} & C(\psi;t+1,T)b_g - \psi\xi_g^*(1) + \xi_g^*(\psi + B(\psi;t+1,T)d_z^* + C(\psi;t+1,T)d_g^* \\
& + D(\psi;t+1,T)d_b^*),
\end{aligned}
$$

$$
\begin{aligned}
D(\psi;t,T) = {} & D(\psi;t+1,T)b_b - \psi\xi_b^*(-1) + \xi_b^*(-\psi + B(\psi;t+1,T)e_z^* + C(\psi;t+1,T)e_g^* \\
& + D(\psi;t+1,T)e_b^*),
\end{aligned}
$$

with terminal conditions $A(\psi;T,T) = B(\psi;T,T) = C(\psi;T,T) = D(\psi;T,T) = 0$. Finally, using the inverse Fourier transformation method employed by Heston and Nandi (2000) to transform the moment generating function, the closed-form solution for European call option value can be priced as follows

$$
C\left(S_t, K, T, h_{z,t+1}^{\mathbb{Q}}, h_{g,t+1}^{\mathbb{Q}}, h_{b,t+1}^{\mathbb{Q}}\right) = S_t P_{1,t} - K\exp\left[-r_t(T-t)\right] P_{2,t}, \qquad (20)
$$

where

$$
P_{1,t} = \frac{1}{2} + \frac{1}{\pi}\exp\left[-r_t(T-t)\right] \int_0^\infty Re\left[\frac{K^{-i\phi} f^{\mathbb{Q}}(\phi+1,t,T)}{i\phi S_t}\right],
$$

$$
P_{2,t} = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left[\frac{K^{-i\phi} f^{\mathbb{Q}}(\phi,t,T)}{i\phi}\right],
$$

C denotes the option value function, S_t represents the asset price, K is the strike price of the option contract, T denotes the expiration date of the contract, $Re[\cdot]$ represents the real part of a complex number, and r_t is the risk-free rate.
Proof. *See Appendix D.* ■

2.5 Good and bad jump variance risk premium

To calculate the BJVRP and GJVRP, we refer to the decomposition process proposed by Chang et al. (2019) and decompose the quadratic variation (QV) of asset returns into a real measure and risk-neutral measure as follows

$$QV^{\mathbb{P}}_{t,t+1} = CV^{\mathbb{P}}_{t,t+1} + GJV^{\mathbb{P}}_{t,t+1} + BJV^{\mathbb{P}}_{t,t+1}, \tag{21}$$

$$QV^{\mathbb{Q}}_{t,t+1} = CV^{\mathbb{Q}}_{t,t+1} + GJV^{\mathbb{Q}}_{t,t+1} + BJV^{\mathbb{Q}}_{t,t+1}, \tag{22}$$

where $CV^{\mathbb{P}}_{t,t+1} = E^{\mathbb{P}}_t[z^2_{t+1}] = h_{z,t+1}$ and $CV^{\mathbb{Q}}_{t,t+1} = E^{\mathbb{Q}}_t[z^{*2}_{t+1}] = h^*_{z,t+1}$. The mathematical expressions for $GJV^{\mathbb{P}}_{t,t+1}$, $GJV^{\mathbb{Q}}_{t,t+1}$, $BJV^{\mathbb{P}}_{t,t+1}$, $BJV^{\mathbb{Q}}_{t,t+1}$ are as follows

$$GJV^{\mathbb{P}}_{t,t+1} = E^{\mathbb{P}}_t\left[\sum_{j=0}^{n_{g,t+1}} (x^j_{g,t+1})^2\right] - E^{\mathbb{P}}_t\left[\sum_{j=0}^{n_{g,t+1}} x^j_{g,t+1}\right]^2 = \frac{2}{\mu_g^2} h_{g,t+1}, \tag{23}$$

$$GJV^{\mathbb{Q}}_{t,t+1} = E^{\mathbb{Q}}_t\left[\sum_{j=0}^{n_{g,t+1}} (x^{*j}_{g,t+1})^2\right] - E^{\mathbb{Q}}_t\left[\sum_{j=0}^{n_{g,t+1}} x^{*j}_{g,t+1}\right]^2 = \frac{2}{\mu_g^{*2}} h^*_{g,t+1}, \tag{24}$$

$$BJV^{\mathbb{P}}_{t,t+1} = E^{\mathbb{P}}_t\left[\sum_{j=0}^{n_{b,t+1}} (x^j_{b,t+1})^2\right] - E^{\mathbb{P}}_t\left[\sum_{j=0}^{n_{b,t+1}} x^j_{b,t+1}\right]^2 = \frac{2}{\mu_b^2} h_{b,t+1}, \tag{25}$$

$$BJV^{\mathbb{Q}}_{t,t+1} = E^{\mathbb{Q}}_t\left[\sum_{j=0}^{n_{b,t+1}} (x^{*j}_{b,t+1})^2\right] - E^{\mathbb{Q}}_t\left[\sum_{j=0}^{n_{b,t+1}} x^{*j}_{b,t+1}\right]^2 = \frac{2}{\mu_b^{*2}} h^*_{b,t+1}. \tag{26}$$

The VRP can be characterized as the discrepancy between the QV under the real measure and that under the risk-neutral measure as follows:

$$VRP_{t,t+1} = E^{\mathbb{P}}_t\left[QV^{\mathbb{P}}_{t,t+1}\right] - E^{\mathbb{Q}}_t\left[QV^{\mathbb{Q}}_{t,t+1}\right]. \tag{27}$$

Using Equations (3.21)-(3.24), we can decompose VRP into

$$\begin{aligned} VRP_{t,t+1} &= E^{\mathbb{P}}_t\left[CV^{\mathbb{P}}_{t,t+1} + GJV^{\mathbb{P}}_{t,t+1} + BJV^{\mathbb{P}}_{t,t+1}\right] - E^{\mathbb{Q}}_t\left[CV^{\mathbb{Q}}_{t,t+1} + GJV^{\mathbb{Q}}_{t,t+1} + BJV^{\mathbb{Q}}_{t,t+1}\right] \\ &= \left(E^{\mathbb{P}}_t[CV^{\mathbb{P}}_{t,t+1}] - E^{\mathbb{Q}}_t[CV^{\mathbb{Q}}_{t,t+1}]\right) + \left(E^{\mathbb{P}}_t[GJV^{\mathbb{P}}_{t,t+1}] - E^{\mathbb{Q}}_t[GJV^{\mathbb{Q}}_{t,t+1}]\right) + \\ &\left(E^{\mathbb{P}}_t[BJV^{\mathbb{P}}_{t,t+1}] - E^{\mathbb{Q}}_t[BJV^{\mathbb{Q}}_{t,t+1}]\right) \\ &= CVRP_{t,t+1} + GJVRP_{t,t+1} + BJVRP_{t,t+1}. \end{aligned} \tag{28}$$

If the JVRP is assumed to be the sum of the GJVRP and BJVRP, Equation (28) becomes Equation (32) in Chang et al. (2019). We then aim to decompose the total VRP into the

CVRP, GJVRP, and BJVRP. Therefore, the BJVRP and GJVRP are given by

$$GJVRP_{t,t+1} = E_t^{\mathbb{P}}[GJV_{t,t+1}^{\mathbb{P}}] - E_t^{\mathbb{Q}}[GJV_{t,t+1}^{\mathbb{Q}}] = \frac{2}{\mu_g^2} h_{g,t+1} - \frac{2}{\mu_g^{*2}} h_{g,t+1}^*, \tag{29}$$

$$BJVRP_{t,t+1} = E_t^{\mathbb{P}}[BJV_{t,t+1}^{\mathbb{P}}] - E_t^{\mathbb{Q}}[BJV_{t,t+1}^{\mathbb{Q}}] = \frac{2}{\mu_b^2} h_{b,t+1} - \frac{2}{\mu_b^{*2}} h_{b,t+1}^*. \tag{30}$$

3 Data and model estimation

3.1 Data

Data on index returns are obtained from the Center for Research in Security Prices (CRSP), while the daily time series of three-month Treasury bills serve as the proxy for the risk-free rate. Option quotes for S&P 500 index puts and calls are sourced from OptionMetrics, encompassing the timeframe from January 1996 through December 2020. In filtering option data to ensure a focus on highly liquid out-of-the-money options, we adhere to the methodology outlined by Bégin, Dorion, and Gauthier (2020)[1]. Descriptive statistics for the option data, segmented by moneyness and maturity, are presented in Table 1.

3.2 Model estimation

3.2.1 Joint MLE

The joint MLE is crucial to the research of this paper, which means using both asset return and option data to estimate parameters simultaneously. In fact, the precision of estimating the risk premium parameters using only asset return data is relatively poor, but these parameters play a role as a bridge between true measure and risk-neutral measure in pricing kernels (Chernov and Ghysels (2000), Eraker (2004), Christoffersen, Jacobs, and Ornthanalai (2012), Ornthanalai (2014)). Merton (1976) mentions that, in the absence of jumps, deep out-of-the-money option data have little value due to the short maturity of options. This implies that when jumps occur, these deep out-of-the-money options can improve the ability to estimate jump probabilities. Therefore, the richness of option data can help investors extract the characteristics of risk premium more accurately.

We adopt the parameter estimation method proposed by Christoffersen, Jacobs, and Ornthanalai (2012), Ornthanalai (2014), and Chang et al. (2019), which employ the maximum weighted joint log-likelihood function, expressed as follows:

[1]Our option data filtering process is based on the steps outlined by Bégin, Dorion, and Gauthier (2020) in the online appendix under the section 'OA.F More on the Dataset'.

Table 1: **Description of the S&P500 index and option data**

Note: Table 1 presents summary statistics for the daily returns of the S&P 500 index and the corresponding option prices. Panel A provides return statistics for the S&P 500 index. Panels B through D offer summary statistics for the volume of option contracts, their prices, and the implied volatility, organized by moneyness (K/S) and days-to-maturity (M), where S denotes the index level and K is the strike price. The scope of our analysis is limited to out-of-the-money European put and call options on the S&P 500 index. The index data and option data are procured from CRSP and OptionMetrics, respectively. Recorded every Wednesday, option prices from actively traded options are included, with put prices being converted to call prices in accordance with put-call parity. Implied volatilities are determined using the Black-Scholes model. To filter out illiquid options, we follow the procedure described by Bégin, Dorion, and Gauthier (2020). The period of our data sample extends from January 1996 through December 2020.

Panel A: S&P500 index returns

N	Minimum	25%	50%	75%	Maximum	Mean	SD
6,295	-5.54%	-0.21%	0.03%	0.26%	4.76%	0.01%	0.53%

Panel B: Number of option contracts.

	$M \leq 30$	$30 < M \leq 90$	$90 < M \leq 180$	$180 < M \leq 250$	$M > 250$	All
$0.80 < K/S \leq 0.85$	3,241	4,249	2,681	1,107	1,216	12,494
$0.85 < K/S \leq 0.90$	3,912	4,619	2,869	1,191	1,437	14,028
$0.90 < K/S \leq 0.95$	4,195	4,791	2,980	1,293	1,576	14,835
$0.95 < K/S \leq 1.00$	4,146	4,801	3,006	1,332	1,738	15,023
$1.00 < K/S \leq 1.05$	4,148	4,701	2,708	978	1,242	13,777
$1.05 < K/S \leq 1.10$	3,169	4,620	2,809	1,027	1,243	12,868
$1.10 < K/S \leq 1.15$	976	2,420	2,169	989	1,100	7,654
$1.15 < K/S \leq 1.20$	269	820	1,062	733	912	3,796
All	24,056	31,021	20,284	8,650	10,464	94,475

Panel C: Average option prices.

	$M \leq 30$	$30 < M \leq 90$	$90 < M \leq 180$	$180 < M \leq 250$	$M > 250$	All
$0.80 < K/S \leq 0.85$	1.55	5.28	17.85	31.30	43.22	13.01
$0.85 < K/S \leq 0.90$	2.67	9.09	27.29	42.57	57.12	18.78
$0.90 < K/S \leq 0.95$	5.89	16.57	41.47	57.74	74.51	28.29
$0.95 < K/S \leq 1.00$	16.63	33.58	66.47	84.55	102.72	48.00
$1.00 < K/S \leq 1.05$	12.94	25.21	55.25	78.21	99.49	37.88
$1.05 < K/S \leq 1.10$	2.18	6.63	21.00	38.61	54.38	15.84
$1.10 < K/S \leq 1.15$	1.41	3.26	9.87	18.78	29.26	10.64
$1.15 < K/S \leq 1.20$	1.29	2.81	7.06	10.55	15.91	8.53
All	7.13	14.97	33.87	47.98	63.88	25.47

Panel D: Average implied volatility.

	$M \leq 30$	$30 < M \leq 90$	$90 < M \leq 180$	$180 < M \leq 250$	$M > 250$	All
$0.80 < K/S \leq 0.85$	0.3905	0.2995	0.2692	0.2600	0.2491	0.3082
$0.85 < K/S \leq 0.90$	0.3130	0.2580	0.2426	0.2372	0.2294	0.2655
$0.90 < K/S \leq 0.95$	0.2399	0.2207	0.2175	0.2182	0.2132	0.2245
$0.95 < K/S \leq 1.00$	0.1799	0.1847	0.1914	0.1979	0.1960	0.1872
$1.00 < K/S \leq 1.05$	0.1361	0.1506	0.1651	0.1761	0.1808	0.1536
$1.05 < K/S \leq 1.10$	0.1543	0.1382	0.1455	0.1593	0.1615	0.1477
$1.10 < K/S \leq 1.15$	0.2202	0.1650	0.1473	0.1509	0.1517	0.1633
$1.15 < K/S \leq 1.20$	0.2972	0.2083	0.1696	0.1523	0.1468	0.1782
All	0.2324	0.2039	0.1970	0.1980	0.1945	0.2081

$$\max_{\Theta}\left[\frac{T+N}{2}\left(\frac{L_{return}(\Theta)}{T}+\frac{L_{option}(\Theta)}{N}\right)\right], \tag{31}$$

where T is the number of asset returns, N is the number of option data, Θ is the model parameter set, and L_{return} and L_{option} are the log-likelihood functions of asset returns and option data, respectively. Obtaining L_{return} is more complicated; thus, we first introduce the calculation of L_{option}.

Before the L_{option} is calculated, the option pricing error must be estimated. To facilitate this calculation, many studies use vega-weighted pricing errors as proposed by Carr and Wu (2007), Trolle and Schwartz (2009), and Chang et al. (2019). These can be expressed as follows:

$$e_k = \frac{C_k^{MODEL} - C_k^{MKT}}{V_k^{MKT}}, \tag{32}$$

where the kth option C_k^{MODEL} represents the option value calculated with the model, C_k^{MKT} represents the market value, and V_k^{MKT} represents the Black–Scholes vega value. If pricing errors are assumed to be normally distributed ($e_k \sim N(0,\sigma_e^2)$), then L_{option} can be expressed as

$$L_{option}(\Theta) = -\frac{1}{2}\sum_{k=1}^{N}\left(\log(2\pi\sigma_e^2)+\frac{e_k^2}{\sigma_e^2}\right), \tag{33}$$

where σ_e is the sample standard deviation estimation of $\{e_k\}_{k=1}^{N}$.

3.2.2 Particle filter algorithm

For the derivation of L_{return}, in traditional GARCH processes, the noise term can be determined by examining both the return and the initial variance. However, with the introduction of jump factors, distinguishing the proportion of the noise term originating from the stochastic continuous term versus that from the jump term based solely on the return and the initial variance is challenging (Durham, Geweke, and Ghosh (2015)). Thus, even if the returns at time point t are observable, ascertaining the values of $h_{z,t}$, $h_{g,t}$, and $h_{b,t}$ is unfeasible due to the indeterminacy of the values of z_t, $y_{g,t}$, and $y_{b,t}$ respectively. Therefore, we use the particle filter algorithm to obtain z_t, $y_{g,t}$, and $y_{b,t}$ at time t. This, in turn, allows us to compute L_{return}. Previous studies have typically used only two latent variables, namely normal and jump innovations. By contrast, our model for asset return dynamics has three latent variables, specifically normal, good jump, and bad jump innovations. To handle three latent variables in the likelihood function, we use an extension of the novel particle filtering approach called the sequential importance resampling filtering method, which was introduced by Bégin, Dorion, and Gauthier (2020)

Appendix E details the particle filter algorithm.

3.2.3 Model performance

Given that our model allows for the simultaneous inclusion of both good jump and bad jump innovations, we seek to investigate the performance implications of incorporating different jump components into the variance process. We consider four nested models. Model 1 (GARCH with no jump) is based on the affine GARCH framework. This means that neither good jump nor bad jump innovations can influence the variance of the normal term, good jump intensity, or bad jump intensity in the subsequent period ($d_z = d_g = d_g = e_z = e_g = e_d = 0$). In the model, only the residuals stemming from the normal distribution exert influence. Model 2 (GARCH with good jump) is an extension of Model 1; specifically, model 2 includes the effect of good jump innovation ($e_z = e_g = e_d = 0$). Model 3 (GARCH with bad jump) is also an extension of Model 1; specifically, model 3 incorporates only the influence of bad jump innovation ($d_z = d_g = d_g = 0$). Model 4 (GARCH with good and bad jumps) is the most comprehensive configuration; specifically, both good jump and bad jump innovations are included in the variance process. The performance of these models is primarily evaluated in terms of the weighted likelihood and the root mean square error (RMSE) metric, which is defined as follows

$$RMSE = \sqrt{\frac{1}{N}\sum_{k=1}^{N}(C_k^{MODEL} - C_k^{MKT})^2}. \tag{34}$$

Table 2 displays the outcomes of our model estimations. Model 4 has the best performance, followed by Model 3, Model 2, and Model 1. These results reveal the value of incorporating jump components into the variance process, and bad jumps are more influential than good ones. This observation is consonant with the phenomenon where decreases in asset returns during bad jumps tend to be greater than increases in asset returns during good jumps. This also suggests that incorporating a bad jump component within the variance process more effectively captures the changes in the variance for the subsequent period. Finally, we examined the proportion of variance attributable to each risk factor relative to the total variance. Consistent with findings in the literature, the variance from the normal component has the largest share followed by the variance due to bad jumps, and the variance attributable to good jumps occupies the smallest share. This implies that the variance from bad jumps almost completely dominates the variance stemming from the jump component.

A deeper analysis of Model 4 yields additional findings. First, when assessing the risk premium's directional estimates, denoted λ, we find that $\lambda_z > 0$, $\lambda_g < 0$, and $\lambda_b > 0$, consistent with our initial hypotheses. For the good jump innovation, investors are clearly willing to incur a marginal risk premium to benefit from possible good jumps, albeit to a moderate degree. Furthermore, observing the asymmetry parameter c in the model, we find a positive c in good jump intensity, $h_{g,t}$, and a negative c in the variance of normal innovation, $h_{z,t}$, and in bad jump intensity, $h_{b,t}$. According to previous studies, $c < 0$ suggests a negative correlation between asset returns and the variance of the risk factor. Hence, a negative c was observed in $h_{z,t}$ and $h_{b,t}$. However, with positive asset returns, the likelihood of a good jump occurrence increases. Thus, asset returns are positively correlated with good jump intensity, leading to the positive estimate for c.

Furthermore, the parameters d and e represent the clustering effects of the subsequent variance process attributed to good and bad jump innovations, respectively. Good jumps affect good jump intensity more than do the variance of normal innovation or bad jump intensity. That is, a good jump event augments the likelihood of subsequent good jumps. Analogously, bad jumps have a greater impact on bad jump intensity than the do the variance of normal innovation or good jump intensity. This pattern indicates that following a bad jump event, the probability of encountering another in the subsequent timeframe increases. Thus, an evaluation of the magnitude of the good jump $(1/\mu_g)$ and the bad jump $(1/\mu_b)$ reveals that good jumps are more intense than bad ones. This observation is consistent with economic logic; downturns in asset returns are typically more severe than upswings. Because Model 4 has the best performance, we concentrate on the parameter set of Model 4 in the subsequent sections of this paper.

Table 2: **Model parameter estimation by joint MLE**

Note: Note: We employ the joint MLE using daily returns on the S&P 500, along with out-of-the-money options available on Wednesdays, to estimate four distinct models. Specifically, Model 1 (GARCH w/ no jump) models a variance process without including any jump innovations. Model 2 (GARCH w/ good jump) models a variance process with only good jump innovations. Model 3 (GARCH w/ bad jump) models a variance process with only bad jump innovations. Finally, Model 4 (GARCH w/ good and bad jump) models a variance process with both good and bad jump innovations. Our data was for the 1996 to 2020 period. Parameter estimation is conducted utilizing the *fmincon* function in MATLAB, with robust standard errors calculated from the outer product of the gradient evaluated at the optimal parameter estimates, following the methodology outlined by Newey and McFadden (1994). Columns labeled "Normal" depict parameter estimates for the normal component. Those labeled "Good Jump" present estimates associated with good jump innovations, whereas those labelled Bad Jump" display estimates for bad jump innovations. The "Weighted likelihood" presents the values of the weighted joint log-likelihood. "RMSE" denotes the root mean square error values. "Average Volatility" denote the mean of daily return volatilities expressed in annualized terms. Both "Average Skewness" and "Average Kurtosis" refer to the mean daily skewness and kurtosis, respectively. "Percent of variance (%)" denotes the proportion of variance attributable to each risk factor relative to the total variance.

	Model 1 (GARCH w/ no jump)			Model 2 (GARCH w/ good jump)			Model 3 (GARCH w/ bad jump)			Model 4 (GARCH w/ good and bad jump)		
	Normal	Good Jump	Bad Jump	Normal	Good Jump	Bad Jump	Normal	Good Jump	Bad Jump	Normal	Good Jump	Bad Jump
λ	2.85E+00	-5.48E-06	4.20E-03	3.22E+00	-5.49E-06	5.30E-03	3.49E+00	-4.16E-06	4.00E-03	2.18E+00	-5.45E-06	4.10E-03
	(2.00E-22)	(3.50E-23)	(2.71E-22)	(1.31E-05)	(2.37E-05)	(3.60E-06)	(2.22E-03)	(1.17E-04)	(2.10E-03)	(3.22E-07)	(8.41E-09)	(8.23E-09)
w	2.64E-06	7.67E-04	-2.80E-03	2.33E-06	1.40E-03	-3.70E-03	2.70E-06	-1.97E-04	-3.20E-03	2.17E-06	-1.60E-03	-3.20E-03
	(1.33E-13)	(5.21E-22)	(2.80E-22)	(2.34E-05)	(6.49E-10)	(9.60E-06)	(3.09E-04)	(9.57E-04)	(7.67E-04)	(3.06E-09)	(5.23E-08)	(2.44E-08)
b	8.79E-01	8.54E-01	9.06E-01	8.80E-01	8.70E-01	9.35E-01	8.86E-01	9.34E-01	9.89E-01	8.73E-01	8.52E-01	9.84E-01
	(3.24E-20)	(4.00E-23)	(1.91E-21)	(7.37E-04)	(2.87E-08)	(3.97E-05)	(4.68E-03)	(6.64E-03)	(1.81E-03)	(5.27E-08)	(1.40E-08)	(1.94E-07)
a	1.79E-06	1.33E-03	2.49E-03	1.73E-06	1.00E-03	2.50E-03	1.72E-06	1.50E-03	2.10E-03	2.07E-06	2.30E-03	2.60E-03
	(8.10E-12)	(1.20E-22)	(8.01E-23)	(1.50E-04)	(1.32E-04)	(1.70E-04)	(1.51E-03)	(6.15E-03)	(2.83E-03)	(1.38E-08)	(4.51E-08)	(8.84E-09)
c	1.13E+02	-1.04E+02	1.20E+02	1.16E+02	-9.56E+01	1.21E+02	1.05E+02	-1.19E+02	1.18E+02	1.17E+02	-1.04E+02	9.04E+01
	(1.27E-20)	(4.84E-22)	(9.61E-22)	(3.25E-05)	(2.12E-04)	(1.37E-04)	(1.55E-03)	(4.64E-03)	(1.76E-03)	(7.84E-09)	(1.18E-08)	(1.47E-07)
d				3.86E-06	6.70E-03	2.86E-06				3.95E-06	7.10E-03	1.97E-06
				(5.92E-12)	(2.12E-11)	(4.35E-12)				(2.61E-08)	(6.84E-08)	(7.71E-09)
e							8.18E-06	2.90E-05	3.80E-03	8.02E-06	2.56E-05	5.40E-03
							(6.66E-04)	(6.06E-04)	(1.72E-03)	(8.80E-09)	(3.03E-09)	(2.49E-08)
ϕ	9.57E-01			9.57E-01			9.59E-01			9.55E-01		
	(1.02E-22)			(3.63E-11)			(4.97E-03)			(1.54E-15)		
μ_g		6.89E+01			8.29E+01			7.83E+01			4.77E+01	
		(2.24E-21)			(1.95E-04)			(2.95E-03)			(4.58E-08)	
μ_b			4.14E+01			2.82E+01			4.65E+01			3.91E+01
			(4.64E-21)			(1.93E-06)			(1.74E-03)			(5.83E-09)
Properties												
Weighted lilikehood	263202			269090			270004			272563		
From returns	200020			200878			198965			201260		
From options	63182			68212			71039			71303		
RMSE	21.4514			19.9152			17.6958			16.7870		
Average Volatility	0.1139			0.1227			0.1268			0.1269		
Average Skewness	-0.0566			-1.0215			-0.1384			-0.6107		
Average Kurtosis	20.9280			35.0281			16.0866			24.7356		
Percent of variance (%)	71.26	11.37	17.37	57.76	8.74	33.50	60.06	10.44	29.50	54.31	6.77	38.92

In Figure 2, we present daily time series data on S&P500 returns, normal innovations, good jump innovations, and bad jump innovations. Notable financial episodes, including the 1998 European Debt Crisis, the 2000-2002 Dot-com Bubble, the 2007-2008 Financial Crisis, the 2010-2011 European Debt Crisis, and the 2018-2019 US-China Trade War, are clearly discernible in the estimated good and bad jump components of our model.

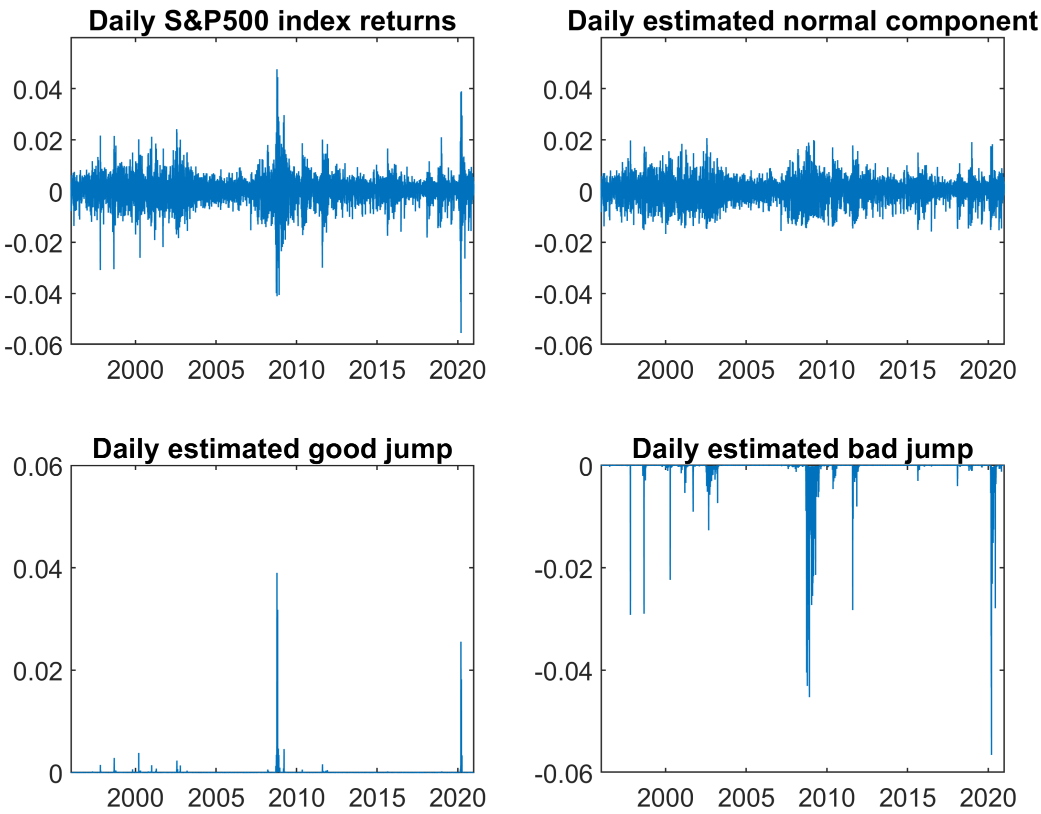

Figure 2: **Daily returns, estimated daily normal component, good jump innovation, and bad jump innovation on the S&P500 index**
Note: In addition to daily returns on the S&P 500 index, based on Model 4's estimates, we plot the estimated daily normal component, the estimated daily good jump innovation, and the estimated daily bad jump innovation.

Figure 3 showcases the monthly time series for the realized variance of the normal component, good jump intensity, and bad jump intensity. We determine the monthly values by aggregating the daily data within each month. During critical financial events, we observe pronounced fluctuations in the variances of these three risk dimensions.

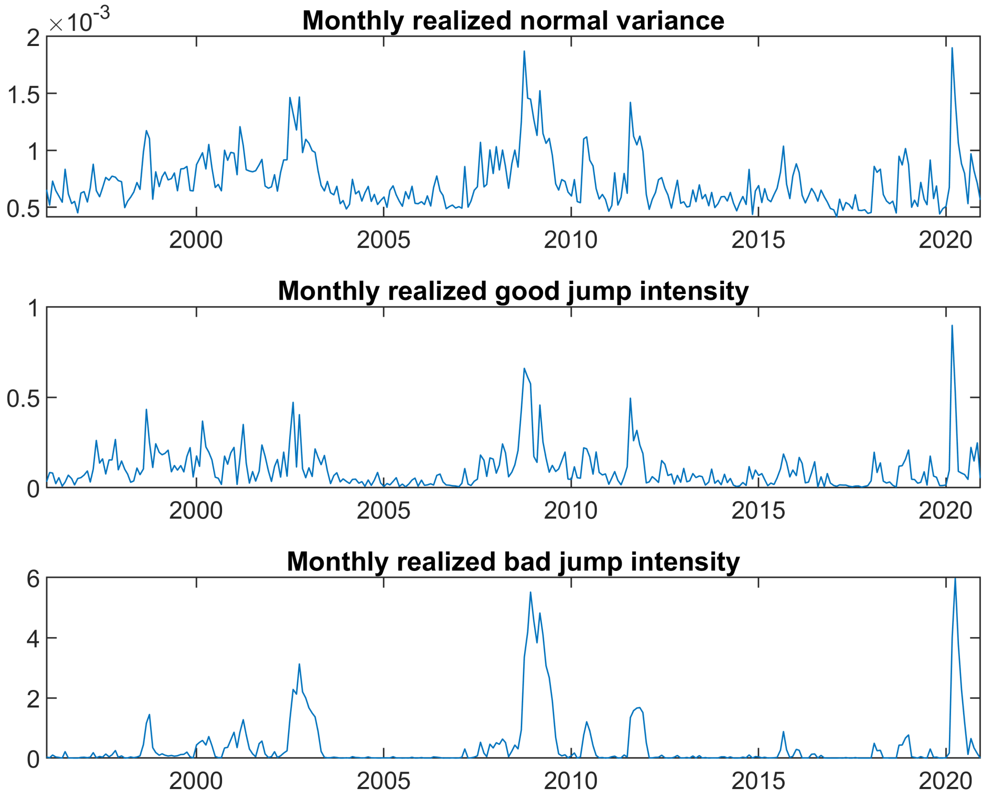

Figure 3: **Monthly time series for the realized variance of normal component, good jump intensity, and bad jump intensity**
Note: Based on the results of Model 4's estimation, we plot the monthly realized variance of the normal component, good jump intensity, and bad jump intensity. The monthly realized variance is defined as the cumulative sum of daily realized variances for that month.

On the basis of Model 4's estimates, we can now identify several variables of interest. Table 3 provides a monthly statistical summary of various VRPs. Notably, the VRP is the sum of the CVRP and JVRP; the JVRP can be further divided into the GJVRP and BJVRP. In our measure transformation, the variance under the $\mathbb{Q}$-measure consistently exceeds that of the $\mathbb{P}$-measure, leading all of our risk variance premiums to be negative. In particular, an examination of CVRP, GJVRP, and BJVRP reveals that the absolute value of BJVRP is greater than the other two VRPs, indicating that when negative news causes asset prices to decline, investors are more fearful of future uncertainties.

Figure 4 depicts the monthly time series for the CVRP, the GJVRP, and the BJVRP. Similar to the aforementioned findings, during significant financial events, all three VRPs have a pronounced downturn. As observed in Figure 3, the BJVRP has the steepest drop, whereas the decrease in the GJVRP is the smallest. This trend is consistent with investor fears leading to a significantly larger increase in the bad jump intensity, as captured in the options' implied volatility, than the good jump intensity.

Table 3: **Summary statistics for monthly VRPs**

Note: We present monthly summary statistics of the VRP, JVRP, CVRP, GJVRP, and BJVRP. The monthly VRP is the sum of the daily VRPs for that month. The upper part of the table presents the descriptive statistics for each type of VRP. The lower part of the table displays the correlation coefficients between these VRPs.

	Min	Max	Mean	SD	AR(1)
VRP	-2.10E-03	-1.94E-05	-1.71E-04	3.23E-04	0.88
JVRP	-2.00E-03	-7.64E-08	-1.37E-04	3.14E-04	0.88
CJVRP	-8.86E-05	-1.93E-05	-3.40E-05	1.11E-05	0.67
GJVRP	-2.98E-07	-1.16E-09	-3.43E-08	3.80E-08	0.55
BJVRP	-2.00E-03	-7.47E-08	-1.37E-04	3.14E-04	0.88
	VRP	JVRP	CVRP	GJVRP	BJVRP
VRP	1	0.9998	0.8050	0.6937	0.9998
JVRP	0.9998	1	0.7923	0.6841	0.9999
CVRP	0.8050	0.7923	1	0.8297	0.7923
GJVRP	0.6937	0.6841	0.8297	1	0.6840
BJVRP	0.9998	0.9999	0.7923	0.6840	1

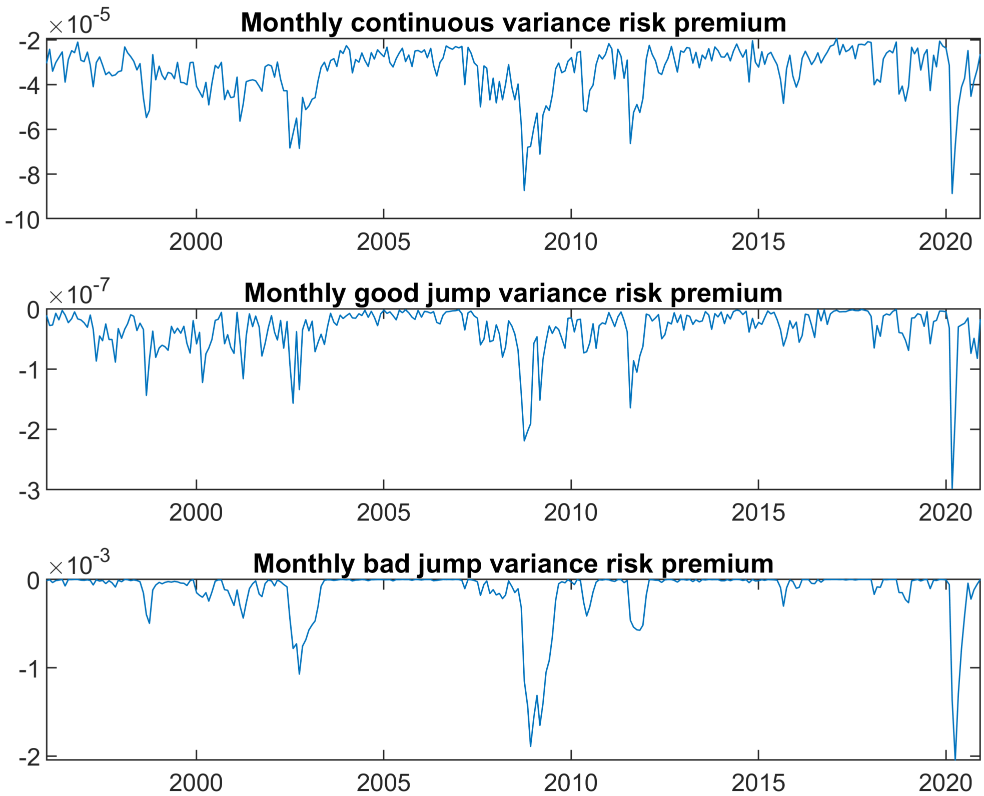

Figure 4: **Monthly time series for CVRP, GJVRP, and BJVRP**

Note: Based on Model 4's estimates, we plot the monthly CVRP, GJVRP, and BJVRP. The monthly VRP is defined as the sum of all daily VRPs for that month.

4 Empirical results

In this section, we further examine whether various VRPs serve as priced risk factors and whether they can predict future asset returns. We first conduct cross-sectional regressions to verify whether the various variance risks act as priced factors. Specifically, the cross-sectional regression model is as follows:

$$r_t = \beta\gamma + \beta f_t + u_t, \tag{35}$$

where r_t represents the equities portfolio's excess return on month t[2], f_t denotes factors including VRPs and Fama-French three factors, γ is the vector of risk premium for f_t, and u_t represents idiosyncratic errors.

In the cross-sectional regression, we primarily adopt two prevalent methods to estimate the risk premium (γ). First, we integrate each VRP with the capital asset pricing model (CAPM) factor and the Fama–French three-factor model (FF3), using the Fama–MacBeth regression to estimate the risk premium for each VRP. The second method of estimating the risk premium for each VRP is based on the three-pass regression approach introduced by Giglio and Xiu (2021). The virtue of the three-pass regression is that it possesses theoretical support suggesting that its estimated risk premiums exhibit robust statistical properties, even if important explanatory variables are omitted.

Table 4 presents the results of risk premium estimations using both the Fama-MacBeth regression and the three-pass regression methods. We note that regardless of the estimation approach, the risk premium for VRP is statistically significant. Further dissecting VRP into its components, CVRP and JVRP, both factors exhibit significant risk premiums as well. However, the risk premium of JVRP is more aligned with that of VRP. Delving deeper into the constituents of JVRP, namely GJVRP and BJVRP, both factors are statistically significant at comparable levels, with BJVRP's risk premium closely mirroring that of JVRP. As a result, we deduce that the pricing capability of VRP is predominantly influenced by BJVRP. Furthermore, when observing the three underlying variance risk premiums, CVRP, GJVRP, and BJVRP, the risk premium associated with BJVRP is the most pronounced. It means that BJVRP captures investors' anticipatory negative views on the market. This negative sentiment leads BJVRP to bring about a higher expected stock return, which is also reflected in the expected returns of both JVRP and VRP.

[2] Following the approach of Giglio and Xiu (2021), we use 202 U.S. equities portfolios, which include 17 industry portfolios, 25 portfolios sorted by size and book-to-market ratio, 25 portfolios sorted by size and variance, 35 portfolios sorted by size and net issuance, 25 portfolios sorted by operating profitability and investment, 25 portfolios sorted by size and accruals, 25 portfolios sorted by size and momentum, and 25 portfolios sorted by size and beta. The data is sourced from French's website and span the period from 1996 to 2020.

Table 4: **Each VRP and the cross–section of equity portfolios**
Note: We document the cross-sectional relationship between the VRP, JVRP, CVRP, GJVRP, and BJVRP with equity portfolios. The risk premiums for these VRPs are estimated using three distinct methodologies: the Fama-MacBeth regression within the CAPM framework, the Fama-French three-factor model (FF3), and the three-pass regression approach as proposed by Giglio and Xiu (2021). To account for serial correlation, robust standard errors are computed in accordance with Newey and West (1994), applying optimal lag selection. In the results, 'Coef' signifies the estimated risk premium coefficient, and 't' represents the t-statistic for this estimate.

	with CAPM		with FF3		Three-pass regresion	
	Coef	t	Coef	t	Coef	t
VRP	1.76E-04	3.45	1.62E-04	3.38	3.00E-05	5.75
JVRP	1.71E-04	3.45	1.57E-04	3.37	2.87E-05	5.60
CVRP	6.81E-06	3.34	6.35E-06	3.39	1.30E-06	6.26
GJVRP	1.72E-08	2.61	1.89E-08	3.12	2.56E-09	4.36
BJVRP	1.71E-04	3.45	1.57E-04	3.37	2.87E-05	5.60

In the subsequent phase, we examine whether each variance risk premium possesses return predictability. We conduct the following regression:

$$\frac{1}{h}\sum_{j=1}^{h} r_{t+j} = \alpha_h + \beta_h X_t + u_{t+h}, \tag{36}$$

where h is the regression horizon, r_t denotes a log excess return of S&P500 index on month t, X_t contains the predictor variables including VRP, JVRP, CVRP, GJVRP and BJVRP. From Table 3, we observe a high degree of correlation among the VRPs, thus we employ OLS to estimate future returns based solely on each individual VRP.

Drawing from previous literature on the predictive power of VRPs, they have high predictive ability for time horizons longer than 6 months. To obtain more robust results, we set the regression horizon h to 1, 3, 6, 12, 18, or 24 months. The Newey—West standard errors were used to determine the optimal lag length. Table 5 documents the predictability regression results ranging from 3 to 24 months. Given the substantial numerical differences among the variables, we standardize each variable to facilitate the interpretation of subsequent empirical results.

Table 5 showcases the predictability of returns for each factor. Due to high correlations among our factors, our regression results focus on univariate predictability regressions for each factor. Consistent with existing literature, the coefficient for VRP is statistically significant when predicting returns over periods exceeding six months. Moreover, as the prediction horizon lengthens, the level of significance escalates. When we further scrutinize JVRP and CVRP, we find that the significance level for JVRP almost mirrors that of VRP, whereas CVRP remains insignificant regardless of the prediction horizon. Subsequently, when examining GJVRP and BJVRP, our findings align BJVRP closely

Table 5: **Return predictability of each VRP**

Note: We report the ordinary least squares predictive regression estimates of index return on each VRP. The dependent variables are 1-, 3-, 6-, 12-, 18-, 24-month S&P500 index returns. We standardize each VRP. To account for serial correlation, robust standard errors are computed in accordance with Newey and West (1994), applying optimal lag selection. In the results, 'Coef' signifies the estimated risk premium coefficient, and 't' represents the t-statistic for this estimate, and 'R^2' is the coefficient of determination.

	1			3			6		
	Coef	t	R^2(%)	Coef	t	R^2(%)	Coef	t	R^2(%)
VRP	-0.0718	-0.67	0.52	-0.1459	-1.17	2.15	-0.2266	-3.03	5.22
JVRP	-0.0732	-0.68	0.54	-0.1491	-1.21	2.24	-0.2317	-3.17	5.45
CVRP	-0.0162	-0.15	0.03	-0.0267	-0.19	0.07	-0.0368	-0.32	0.14
GJVRP	0.0027	0.02	0.00	0.0132	0.09	0.02	-0.0144	-0.14	0.02
BJVRP	-0.0732	-0.68	0.54	-0.1491	-1.21	2.24	-0.2317	-3.17	5.46
	12			18			24		
	Coef	t	R^2(%)	Coef	t	R^2(%)	Coef	t	R^2(%)
VRP	-0.2356	-3.39	4.40	-0.2021	-3.17	3.30	-0.2686	-4.03	5.92
JVRP	-0.2439	-3.70	4.70	-0.2092	-3.39	3.52	-0.2749	-4.27	6.19
CVRP	0.0196	0.16	0.03	0.0162	0.14	0.02	-0.0542	-0.46	0.27
GJVRP	0.0093	0.09	0.00	-0.0119	-0.12	0.01	-0.0528	-0.52	0.23
BJVRP	-0.2439	-3.70	4.70	-0.2092	-3.39	3.52	-0.2749	-4.27	6.19

with JVRP, while GJVRP remains statistically insignificant. This suggests that the return predictability of VRP primarily emanates from the jump component rather than the continuous component. Furthermore, it is the bad jump component that dominates the results.

The results for R^2 reveal a peak value at 6% for BJVRP when predicting returns over the subsequent 24 months. Across varying periods, BJVRP's R^2 considerably outpaces those of CVRP and GJVRP. Ultimately, we discern that when predicting returns beyond a six-month horizon, an increase in one standard deviation in BJVRP leads to a decrease in future returns by approximately 25% of a standard deviation.

5 Conclusion

In this paper, we investigate the roles of good jump variance risk and bad jump variance risk in option pricing. Extending the model proposed by Yang (2018), we bifurcate the asymmetric double exponential jump distribution model into one that simultaneously captures both good and bad jumps. Subsequently, we derive the closed-form solutions for option pricing corresponding to this model. With regard to estimation, we investigate the model performance implications of integrating different types of jumps into the variance process. Our findings reveal that the model with both good and bad jumps outperforms the others. Furthermore, the model incorporating solely the bad jump component fares better than its counterpart that only includes the good jump component. Empirically, cross-sectional regression estimates indicate that CVRP, GJVRP, or BJVRP are all price risk factors. However, BJVRP aligns more closely with JVRP and VRP and, compared with CVRP and GJVRP, offers a higher risk premium. Beyond this, in our time-series analysis, BJVRP dominates the results for VRP. When predicting returns for periods extending beyond six months, we observe significant predictive performance. A one-standard-deviation increase in BJVRP correlates with a subsequent decline in future returns by approximately 25% of a standard deviation.

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Declaration of interest

No potential conflict of interest was reported by the authors.

Declaration of generative AI in scientific writing

During the preparation of this work we used ChatGPT4 in order to proofread this paper. After using this tool, we reviewed and edited the content as needed and take full responsibility for the content of the publication.

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Appendix A

Proof of Lemma 1. Recall that z_t, $y_{g,t}$ and $y_{b,t}$ are conditionally indepedent, we have

$$\begin{aligned}
&E^{\mathbb{P}}_{t-1}\left[\exp\left(-\Lambda_Z z_t + \Lambda_{VZ}(z_t^2/h_{z,t}) - \Lambda_G y_{g,t} - \Lambda_B y_{b,t}\right)\right] \\
&\quad = E^{\mathbb{P}}_{t-1}\left[\exp\left(-\Lambda_Z z_t + \Lambda_{VZ}(z_t^2/h_{z,t})\right)\right] E^{\mathbb{P}}_{t-1}\left[\exp\left(-\Lambda_G y_{g,t}\right)\right] E^{\mathbb{P}}_{t-1}\left[\exp\left(-\Lambda_B y_{b,t}\right)\right].
\end{aligned}$$

Also, we have the following properties for the normal variable and the compound Poisson variable

$$E^{\mathbb{P}}_{t-1}\left[\exp\left(-\Lambda_Z z_t + \Lambda_{VZ}(z_t^2/h_{z,t})\right)\right] = \exp\left(-\frac{\log(1-2\Lambda_{VZ})}{2} + \frac{\Lambda_Z^2}{2(1-2\Lambda_{VZ})}h_{z,t}\right),$$

$$E^{\mathbb{P}}_{t-1}\left[\exp\left(-\Lambda_G y_{g,t}\right)\right] = \exp\left(\left(\frac{\mu_g}{\mu_g+\Lambda_G} - 1\right) h_{g,t}\right),$$

$$E^{\mathbb{P}}_{t-1}\left[\exp\left(-\Lambda_B y_{b,t}\right)\right] = \exp\left(\left(\frac{\mu_b}{\mu_b+\Lambda_B} - 1\right) h_{b,t}\right).$$

Using the above formulations, we can accomplish the lemma as follows

$$\begin{aligned}
&E^{\mathbb{P}}_{t-1}\left[\exp\left(-\Lambda_Z z_t + \Lambda_{VZ}(z_t^2/h_{z,t}) - \Lambda_G y_{g,t} - \Lambda_B y_{b,t}\right)\right] \\
&= \exp\left(-\frac{\log(1-2\Lambda_{VZ})}{2} + \frac{\Lambda_Z^2}{2(1-2\Lambda_{VZ})}h_{z,t} + \left(\frac{\mu_g}{\mu_g+\Lambda_G} - 1\right) h_{g,t} + \left(\frac{\mu_b}{\mu_b+\Lambda_B} - 1\right) h_{b,t}\right).
\end{aligned}$$

■

Appendix B

Proof of Proposition 1. If an equivalent martingale measure exists, the expected return of asset on time from $t-1$ to t under measure $\mathbb{Q}$ is the risk-free rate

$$E_{t-1}^{\mathbb{Q}}\left[\frac{S_t}{S_{t-1}}\right] = e^{r_t}.$$

Using the $\mathbb{Q}$-to-$\mathbb{P}$ measure transformation and incorporating the Radon—Nikodým derivative in Equation (10) along with Lemma 1 and the asset dynamics process in Equation (1) into the above equation, we have

$$E_{t-1}^{\mathbb{Q}}\left[\frac{S_t}{S_{t-1}}\right] = E_{t-1}^{\mathbb{P}}\left[\left(\frac{d\mathbb{Q}_t/d\mathbb{P}_t}{d\mathbb{Q}_{t-1}/d\mathbb{P}_{t-1}}\right)\frac{S_t}{S_{t-1}}\exp(-r_t)\right] = 1.$$

Therefore,

$$\begin{aligned}
&E_{t-1}^{\mathbb{Q}}\left[\frac{S_t}{S_{t-1}}\right] = E_{t-1}^{\mathbb{P}}\left[\left(\frac{d\mathbb{Q}_t/d\mathbb{P}_t}{d\mathbb{Q}_{t-1}/d\mathbb{P}_{t-1}}\right)\frac{S_t}{S_{t-1}}\exp(-r_t)\right]\\
&= E_{t-1}^{\mathbb{P}}\left[\exp\left\{-\Lambda_Z z_t + \Lambda_{VZ}\frac{z_t^2}{h_{z,t}} - \Lambda_G y_{g,t} - \Lambda_B y_{b,t} + (\lambda_z - \xi_z(1))\,h_{z,t} + (\lambda_g - \xi_g(1))\,h_{g,t}\right.\right.\\
&+ (\lambda_b - \xi_b(-1))\,h_{b,t} + z_t + y_{g,t} - y_{b,t} + \frac{\log(1-2\Lambda_{VZ})}{2} - \frac{\Lambda_Z^2}{2(1-2\Lambda_{VZ})}h_{z,t} - \left(\frac{\mu_g}{\mu_g+\Lambda_G} - 1\right)h_{g,t}\\
&\left.\left.- \left(\frac{\mu_b}{\mu_b+\Lambda_B} - 1\right)h_{b,t}\right\}\right]\\
&= \exp\left\{(\lambda_z - \xi_z(1))\,h_{z,t} + (\lambda_g - \xi_g(1))\,h_{g,t} + (\lambda_b - \xi_b(-1))\,h_{b,t} + \frac{\log(1-2\Lambda_{VZ})}{2} - \frac{\Lambda_Z^2}{2(1-2\Lambda_{VZ})}h_{z,t}\right.\\
&\left.- \left(\frac{\mu_g}{\mu_g+\Lambda_G} - 1\right)h_{g,t} - \left(\frac{\mu_b}{\mu_b+\Lambda_B} - 1\right)h_{b,t}\right\}\\
&\times E_{t-1}^{\mathbb{P}}\left[\exp\left\{-(\Lambda_Z - 1)z_t + \Lambda_Z\frac{z_t^2}{h_{z,t}} - (\Lambda_G - 1)y_{g,t} - (\Lambda_B + 1)y_{b,t}\right\}\right]\\
&= \exp\left\{(\lambda_z - \xi_z(1))\,h_{z,t} + (\lambda_g - \xi_g(1))\,h_{g,t} + (\lambda_b - \xi_b(-1))\,h_{b,t} + \frac{\log(1-2\Lambda_{VZ})}{2} - \frac{\Lambda_Z^2}{2(1-2\Lambda_{VZ})}h_{z,t}\right.\\
&- \left(\frac{\mu_g}{\mu_g+\Lambda_G} - 1\right)h_{g,t} - \left(\frac{\mu_b}{\mu_b+\Lambda_B} - 1\right)h_{b,t} - \frac{\log(1-2\Lambda_{VZ})}{2} + \frac{(\Lambda_Z-1)^2}{2(1-2\Lambda_{VZ})}h_{z,t}\\
&\left.+ \left(\frac{\mu_g}{\mu_g+(\Lambda_G-1)} - 1\right)h_{g,t} + \left(\frac{\mu_b}{\mu_b+(\Lambda_B+1)} - 1\right)h_{b,t}\right\}\\
&= \exp\left\{\left(\lambda_z - \xi_z(1) + \frac{1-2\Lambda_Z}{2(1-2\Lambda_Z)}\right)h_{z,t} + \left(\lambda_g - \xi_g(1) - \left(\frac{\mu_g}{\mu_g+\Lambda_G} - 1\right)\right.\right.\\
&\left.\left.+ \left(\frac{\mu_g}{\mu_g+(\Lambda_G-1)} - 1\right)\right)h_{g,t} + \left(\lambda_b - \xi_b(-1) - \left(\frac{\mu_b}{\mu_b+\Lambda_B} - 1\right) + \left(\frac{\mu_b}{\mu_b+(\Lambda_B+1)} - 1\right)\right)h_{b,t}\right\}.
\end{aligned}$$

Thus, the discounted stock is a martingale under the probability measure $\mathbb{Q}$ if and only if

$$\left(\lambda_z - \frac{1}{2} + \frac{1-2\Lambda_Z}{2(1-2\Lambda_Z)}\right) h_{z,t} + \left(\lambda_g - \left(\frac{\mu_g}{\mu_g - 1} - 1\right) - \left(\frac{\mu_g}{\mu_g + \Lambda_G} - 1\right) + \left(\frac{\mu_g}{\mu_g + (\Lambda_G - 1)} - 1\right)\right) h_{g,t}$$
$$+ \left(\lambda_b - \left(\frac{\mu_b}{\mu_b + 1} - 1\right) - \left(\frac{\mu_b}{\mu_b + \Lambda_B} - 1\right) + \left(\frac{\mu_b}{\mu_b + (\Lambda_B + 1)} - 1\right)\right) h_{b,t} = 0.$$

Then, we have

$$\lambda_z - \frac{1}{2} + \frac{1-2\Lambda_Z}{2(1-2\Lambda_Z)} = 0,$$

$$\lambda_g - \left(\frac{\mu_g}{\mu_g - 1} - 1\right) - \left(\frac{\mu_g}{\mu_g + \Lambda_G} - 1\right) + \left(\frac{\mu_g}{\mu_g + (\Lambda_G - 1)} - 1\right) = 0,$$

and

$$\lambda_b - \left(\frac{\mu_b}{\mu_b + 1} - 1\right) - \left(\frac{\mu_b}{\mu_b + \Lambda_B} - 1\right) + \left(\frac{\mu_b}{\mu_b + (\Lambda_B + 1)} - 1\right) = 0.$$

■

Appendix C

Proof of Proposition 2. To derive the asset return process under the $\mathbb{Q}$-measure, we first consider the moment generation function of $z_t + y_{g,t} - y_{b,t}$ under the $\mathbb{Q}$-measure as follows

$$
\begin{aligned}
&E_{t-1}^{\mathbb{Q}}\left[\exp\left\{k(z_t + y_{g,t} - y_{b,t})\right\}\right] = E_{t-1}^{\mathbb{P}}\left[\left(\frac{d\mathbb{Q}_t/d\mathbb{P}_t}{d\mathbb{Q}_{t-1}/d\mathbb{P}_{t-1}}\right)\exp\left\{k(z_t + y_{g,t} - y_{b,t})\right\}\right] \\
&= \exp\left\{\frac{\log(1-2\Lambda_{VZ})}{2} - \frac{\Lambda_Z^2}{2(1-2\Lambda_{VZ})}h_{z,t} - \left(\frac{\mu_g}{\mu_g+\Lambda_G}-1\right)h_{g,t} - \left(\frac{\mu_b}{\mu_b+\Lambda_B}-1\right)h_{b,t}\right\} \\
&\times E_{t-1}^{\mathbb{P}}\left[\exp\left\{-(\Lambda_Z - k)z_t + \Lambda_Z\frac{z_t^2}{h_{z,t}} - (\Lambda_G - k)y_{g,t} - (\Lambda_B + k)y_{b,t}\right\}\right] \\
&= \exp\left\{\frac{\log(1-2\Lambda_{VZ})}{2} - \frac{\Lambda_Z^2}{2(1-2\Lambda_{VZ})}h_{z,t} - \left(\frac{\mu_g}{\mu_g+\Lambda_G}-1\right)h_{g,t} - \left(\frac{\mu_b}{\mu_b+\Lambda_B}-1\right)h_{b,t}\right. \\
&\left.-\frac{\log(1-2\Lambda_{VZ})}{2} + \frac{(\Lambda_Z-k)^2}{2(1-2\Lambda_{VZ})}h_{z,t} + \left(\frac{\mu_g}{\mu_g+(\Lambda_G-k)}-1\right)h_{g,t} + \left(\frac{\mu_b}{\mu_b+(\Lambda_B+k)}-1\right)h_{b,t}\right\} \\
&= \exp\left\{\left(-\frac{\Lambda_Z}{1-2\Lambda_{VZ}}k + \frac{1}{2(1-2\Lambda_{VZ})}k^2\right)h_{z,t} + \left(\frac{\mu_g+\Lambda_G}{\mu_g+\Lambda_G-k}-1\right)\frac{\mu_g}{\mu_g+\Lambda_G}h_{g,t}\right. \\
&\left.+\left(\frac{\mu_b+\Lambda_B}{\mu_b+\Lambda_B+k}-1\right)\frac{\mu_b}{\mu_b+\Lambda_B}h_{b,t}\right\} \\
&= \exp\left\{\left(-\frac{\Lambda_Z}{\phi}k + \frac{1}{2\phi}k^2\right)h_{z,t} + \left(\frac{\mu_g^*}{\mu_g^*-k}-1\right)\Pi_g h_{g,t} + \left(\frac{\mu_b^*}{\mu_b^*+k}-1\right)\Pi_b h_{b,t}\right\},
\end{aligned}
$$

where $\phi = 1 - 2\Lambda_{VZ}$, $\mu_g^* = \mu_g + \Lambda_G$, $\Pi_g = \mu_g/\mu_g^*$, $\mu_b^* = \mu_b + \Lambda_B$, $\Pi_b = \mu_b/\mu_b^*$. Thus, we observe that under $\mathbb{Q}$-measure, $z_t \sim N(-\Lambda_Z h_{z,t}^*, h_{z,t}^*)$, $y_{g,t} \sim CPJ(h_{g,t}^*, \mu_g^*)$ and $y_{b,t} \sim CPJ(h_{b,t}^*, \mu_b^*)$ where $h_{z,t}^* = \frac{h_{z,t}}{\phi}$, $h_{g,t}^* = \Pi_g h_{g,t}$ and $h_{b,t}^* = \Pi_b h_{b,t}$. Now, we suppose that $z_t^* = z_t + \Lambda_Z h_{z,t}^* \sim N(0, h_{z,t}^*)$, $y_{g,t}^* \equiv y_{g,t}$ and $y_{b,t}^* \equiv y_{b,t}$. According to the result of Proposition 1, we can derive the return dynamic under $\mathbb{Q}$-measure as follows

$$
\begin{aligned}
&\log\left(\frac{S_t}{S_{t-1}}\right) = r_t + (\lambda_z - \xi_z(1))\,h_{z,t} + (\lambda_g - \xi_g(1))\,h_{g,t} + (\lambda_b - \xi_b(-1))\,h_{b,t} + z_t + y_{g,t} - y_{b,t} \\
&= r_t - \frac{1-2\Lambda_{VZ}}{2\phi}h_{z,t} - \left(\frac{\mu_g^*}{\mu_g^*-1}-1\right)\Pi_g h_{g,t} - \left(\frac{\mu_b^*}{\mu_b^*+1}-1\right)\Pi_b h_{b,t} + z_t + y_{g,t} - y_{b,t} \\
&= r_t - \xi_z^*(1)h_{z,t}^* - \xi_g^*(1)h_{g,t}^* - \xi_b^*(-1)h_{b,t}^* + z_t^* + y_{g,t}^* - y_{b,t}^*,
\end{aligned}
$$

where $\xi_z^*(t) = t^2/2$, $\xi_g^*(t) = \frac{\mu_g^*}{\mu_g^*-t} - 1$ and $\xi_b^*(t) = \frac{\mu_b^*}{\mu_b^*-t} - 1$.

The variance dynamic under $\mathbb{Q}$-measure can be written as

$$
\begin{aligned}
h_{z,t} &= w_z + b_z h_{z,t-1} + \frac{a_z}{h_{z,t-1}} \left(z_{t-1} - c_z h_{z,t-1}\right)^2 + d_z y_{g,t-1} + e_z y_{b,t-1} \\
h^*_{z,t} &= \frac{w_z}{\phi} + b_z h^*_{z,t-1} + \frac{a_z}{h^*_{z,t-1}\phi^2} \left(z^*_{t-1} - \Lambda_Z h^*_{z,t-1} - c_z h^*_{z,t-1}\phi\right)^2 + \frac{d_z}{\phi} y^*_{g,t-1} + \frac{e_z}{\phi} y^*_{b,t-1} \\
h^*_{z,t} &= w^*_z + b_z h^*_{z,t-1} + \frac{a^*_z}{h^*_{z,t-1}} \left(z^*_{t-1} - c^*_z h^*_{z,t-1}\right)^2 + d^*_z y^*_{g,t-1} + e^*_z y^*_{b,t-1},
\end{aligned}
$$

where $w^*_z = w_z/\phi$, $a^*_z = a_z/\phi^2$, $c^*_z = c_z\phi + \Lambda_Z$, $d^*_z = d_z/\phi$, $e^*_z = e_z/\phi$.

Additionally, the dynamic of good jump intensity under $\mathbb{Q}$-measure can be written as

$$
\begin{aligned}
h_{g,t} &= w_g + b_g h_{g,t-1} + \frac{a_g}{h_{z,t-1}} \left(z_{t-1} - c_g h_{z,t-1}\right)^2 + d_g y_{g,t-1} + e_g y_{b,t-1} \\
h^*_{g,t} &= w_g \Pi_g + b_g h^*_{g,t-1} + \frac{a_g \Pi_g}{h_{z,t-1}\phi} \left(z^*_{t-1} - \Lambda_Z h^*_{z,t-1} - c_g h^*_{z,t-1}\phi\right)^2 + d_g \Pi_g y^*_{g,t-1} + e_g \Pi_g y^*_{b,t-1} \\
h^*_{g,t} &= w^*_g + b_g h^*_{g,t-1} + \frac{a^*_g}{h^*_{z,t-1}} \left(z^*_{t-1} - c^*_g h^*_{z,t-1}\right)^2 + d^*_g y^*_{g,t-1} + e^*_g y^*_{b,t-1},
\end{aligned}
$$

where $w^*_g = w_g \Pi_g$, $a^*_g = a_g \Pi_g/\phi$, $c^*_g = c_g\phi + \Lambda_Z$, $d^*_g = d_g \Pi_g$, $e^*_g = e_g \Pi_g$.

Finally, the dynamic of bad jump intensity under $\mathbb{Q}$-measure can be written as

$$
\begin{aligned}
h_{b,t} &= w_b + b_b h_{b,t-1} + \frac{a_b}{h_{z,t-1}} \left(z_{t-1} - c_b h_{z,t-1}\right)^2 + d_b y_{g,t-1} + e_b y_{b,t-1} \\
h^*_{b,t} &= w_b \Pi_b + b_b h^*_{b,t-1} + \frac{a_b \Pi_b}{h_{z,t-1}\phi} \left(z^*_{t-1} - \Lambda_Z h^*_{z,t-1} - c_b h^*_{z,t-1}\phi\right)^2 + d_b \Pi_b y^*_{g,t-1} + e_b \Pi_b y^*_{b,t-1} \\
h^*_{b,t} &= w^*_b + b_b h^*_{b,t-1} + \frac{a^*_b}{h^*_{z,t-1}} \left(z^*_{t-1} - c^*_b h^*_{z,t-1}\right)^2 + d^*_b y^*_{g,t-1} + e^*_b y^*_{b,t-1},
\end{aligned}
$$

where $w^*_b = w_b \Pi_b$, $a^*_b = a_b \Pi_b/\phi$, $c^*_b = c_b\phi + \Lambda_Z$, $d^*_b = d_b \Pi_b$, $e^*_b = e_b \Pi_b$. ■

Appendix D

Proof of Proposition 3. We first assume the moment generating function of the multi-period asset return as follows

$$f^{\mathbb{Q}}(\phi;t,T) \equiv E_t^{\mathbb{Q}}\left[\exp\left(\psi\sum_{j=1}^{T-t}R_{t+j}\right)\right]$$
$$= \exp\left(A(\psi;t,T) + B(\psi;t,T)h^*_{z,t+1} + C(\psi;t,T)h^*_{g,t+1} + D(\psi;t,T)h^*_{b,t+1}\right),$$

where $A(\psi;t,T)$, $B(\psi;t,T)$, $C(\psi;t,T)$ and $D(\psi;t,T)$ are the scalar coefficients. Then, we have

$$f^{\mathbb{Q}}(\phi;t+1,T) = E_{t+1}^{\mathbb{Q}}\left[\exp\left(\psi\sum_{j=2}^{T-t}R_{t+j}\right)\right]$$
$$= \exp\left(A(\psi;t+1,T) + B(\psi;t+1,T)h^*_{z,t+2} + C(\psi;t+1,T)h^*_{g,t+2}\right.$$
$$\left.+D(\psi;t+1,T)h^*_{b,t+2}\right).$$

According to the property of iterated expectation, we have

$$f^{\mathbb{Q}}(\phi;t,T) = E_t^{\mathbb{Q}}\left[\exp\left(\psi\sum_{j=1}^{T-t}R_{t+j}\right)\right] = E_t^{\mathbb{Q}}\left[E_{t+1}^{\mathbb{Q}}\left[\exp\left(\psi\sum_{j=1}^{T-t}R_{t+j}\right)\right]\right]$$
$$= E_t^{\mathbb{Q}}\left[\exp\left(\psi R_{t+1}\right)E_{t+1}^{\mathbb{Q}}\left[\exp\left(\psi\sum_{j=2}^{T-t}R_{t+j}\right)\right]\right]$$
$$= E_t^{\mathbb{Q}}\left[\exp\left(\psi R_{t+1} + A(\psi;t+1,T) + B(\psi;t+1,T)h^*_{z,t+2} + C(\psi;t+1,T)h^*_{g,t+2}\right.\right.$$
$$\left.\left.+D(\psi;t+1,T)h^*_{b,t+2}\right)\right].$$

We then substitute the R_{t+1} and variance process under $\mathbb{Q}$-measure from Equation (15) to (18)

$$f^{\mathbb{Q}}(\phi;t,T) = \exp\Bigg(\psi r_{t+1} - \frac{\psi}{2}h^*_{z,t+1} - \psi\xi^*_g(1)h^*_{g,t+1} - \psi\xi^*_b(-1)h^*_{b,t+1} + A(\psi;t+1,T)$$
$$+B(\psi;t+1,T)(w^*_z + b_z h^*_{z,t+1}) + C(\psi;t+1,T)(w^*_g + b_g h^*_{g,t+1}) + D(\psi;t+1,T)(w^*_b + b_b h^*_{b,t+1})\Big)$$
$$\times E^{\mathbb{Q}}_t\left[\exp\left\{\left(\psi - 2a^*_z c^*_z B(\psi;t+1,T) - 2a^*_g c^*_g C(\psi;t+1,T) - 2a^*_b c^*_b D(\psi;t+1,T)\right) z^*_{t+1}\right.\right.$$
$$+\left(\frac{a^*_z B(\psi;t+1,T)}{h^*_{z,t+1}} + \frac{a^*_g C(\psi;t+1,T)}{h^*_{g,t+1}} + \frac{a^*_b D(\psi;t+1,T)}{h^*_{b,t+1}}\right) z^{*2}_{t+1} + a^*_z c^{*2}_z B(\psi;t+1,T)h^*_{z,t+1}$$
$$\left.\left.+a^*_g c^{*2}_g C(\psi;t+1,T)h^*_{g,t+1} + a^*_b c^{*2}_b D(\psi;t+1,T)h^*_{b,t+1}\right\}\right]$$
$$\times E^{\mathbb{Q}}_t\left[\exp\left\{\left(\psi + d^*_z B(\psi;t+1,T) + d^*_g C(\psi;t+1,T) + d^*_b D(\psi;t+1,T)\right) y^*_{g,t+1}\right\}\right]$$
$$\times E^{\mathbb{Q}}_t\left[\exp\left\{\left(-\psi + e^*_z B(\psi;t+1,T) + e^*_g C(\psi;t+1,T) + e^*_b D(\psi;t+1,T)\right) y^*_{b,t+1}\right\}\right].$$

Recall that we have the following results

$$E^{\mathbb{P}}_{t-1}\left[\exp\left(-\Lambda_Z z_t + \Lambda_{VZ}(z^2_t/h_{z,t})\right)\right] = \exp\left(-\frac{\log(1-2\Lambda_{VZ})}{2} + \frac{\Lambda^2_Z}{2(1-2\Lambda_{VZ})}h_{z,t}\right),$$

$$E^{\mathbb{P}}_{t-1}\left[\exp\left(-\Lambda_G y_{g,t}\right)\right] = \exp\left(\left(\frac{\mu_g}{\mu_g+\Lambda_G} - 1\right)h_{g,t}\right),$$

$$E^{\mathbb{P}}_{t-1}\left[\exp\left(-\Lambda_B y_{b,t}\right)\right] = \exp\left(\left(\frac{\mu_b}{\mu_b+\Lambda_B} - 1\right)h_{b,t}\right).$$

Thus, we can solve the analytical solutions for the scalar coefficients $A(\psi;t,T)$, $B(\psi;t,T)$, $C(\psi;t,T)$ and $D(\psi;t,T)$:

$$A(\psi;t,T) = \psi r_{t+1} + A(\psi;t+1,T) + B(\psi;t+1,T)w^*_z + C(\psi;t+1,T)w^*_g + D(\psi;t+1,T)w^*_b$$
$$-\frac{1}{2}\log\left(1 - 2B(\psi;t+1,T)a^*_z - 2C(\psi;t+1,T)a^*_g - 2D(\psi;t+1,T)a^*_b\right),$$

$$B(\psi;t,T) = -\frac{\psi}{2} + B(\psi;t+1,T)(b_z + a^*_z c^{*2}_z) + C(\psi;t+1,T)a^*_g c^{*2}_g + +D(\psi;t+1,T)a^*_b c^{*2}_b$$
$$+\frac{(\psi - 2B(\psi;t+1,T)a^*_z c^*_z - 2C(\psi;t+1,T)a^*_g c^*_g - 2D(\psi;t+1,T)a^*_b c^*_b)^2}{2(1 - 2B(\psi;t+1,T)a^*_z - 2C(\psi;t+1,T)a^*_g - 2D(\psi;t+1,T)a^*_b)},$$

$$C(\psi;t,T) = C(\psi;t+1,T)b_g - \psi\xi^*_g(1) + \xi^*_g(\psi + B(\psi;t+1,T)d^*_z + C(\psi;t+1,T)d^*_g$$
$$+ D(\psi;t+1,T)d^*_b),$$

$$D(\psi;t,T) = D(\psi;t+1,T)b_b - \psi\xi_b^*(-1) + \xi_b^*(-\psi + B(\psi;t+1,T)e_z^* + C(\psi;t+1,T)e_g^* + D(\psi;t+1,T)e_b^*),$$

with terminal conditions $A(\psi;T,T) = B(\psi;T,T) = C(\psi;T,T) = D(\psi;T,T) = 0$. ■

Appendix E

Our particle filtering approach is inspired by Bégin, Dorion, and Gauthier (2020). They introduce a novel particle filtering algorithm drawing upon the sequential importance resampling method, which accounts for the uncertainties in both the conditional variance of the stochastic continuous term and the variance of the jump frequency. This algorithm can estimate the filtered mean of z_t, $y_{g,t}$, $y_{b,t}$, $h_{z,t}$, $h_{g,t}$, and $h_{b,t}$ simultaneously.

Assume that at time $t-1$, N good jump particles and bad jump particles are represented as $y^{(i)}_{g,1:t-1} = \{y^{(i)}_{g,1}, y^{(i)}_{g,2}, ..., y^{(i)}_{g,t-1}\}, y^{(i)}_{b,1:t-1} = \{y^{(i)}_{b,1}, y^{(i)}_{b,2}, ..., y^{(i)}_{b,t-1}\}, i \in \{1, ..., N\}$, respectively[3]. Given $R_{1:t-1}$, $h_{z,1}$, $h_{g,1}$, and $h_{b,1}$, we can derive $h^{(i)}_{z,t}$, $h^{(i)}_{g,t}$, and $h^{(i)}_{b,t}$ through the subsequent steps:

1. For each $i \in 1, ..., N$, $y^{(i)}_{g,t}$ will be drawn from a compound Poisson distribution assumed by the particle:

$$f\left(\cdot|y^{(i)}_{g,1:t-1}, y^{(i)}_{b,1:t-1}, R_{1:t-1}\right) = f_{GJ}\left(\cdot|\mu_g, h^{(i)}_{g,t}\right),$$

   and $y^{(i)}_{b,t}$ will be drawn from a compound Poisson distribution assumed by the particle:

$$g\left(\cdot|y^{(i)}_{g,1:t-1}, y^{(i)}_{b,1:t-1}, R_{1:t-1}\right) = g_{BJ}\left(\cdot|\mu_b, h^{(i)}_{b,t}\right).$$

2. For each $i \in 1, ..., N$, the importance weight is updated to reflect the likelihood of the simulated good jump and bad jump particle producing the return R_t at time t:

$$\bar{w}^{(i)}_t = f\left(R_t|R_{1:t-1}, y^{(i)}_{g,1:t}, y^{(i)}_{b,1:t}\right) = \frac{1}{\sqrt{2\pi h^{(i)}_{z,t}}}\exp\left\{-\frac{1}{2}\frac{\left(R_t - m^{(i)}_t\right)^2}{h^{(i)}_{z,t}}\right\},$$

   where $m^{(i)}_t = r_t + (\lambda_z - \xi_z(1))\, h^{(i)}_{z,t} + (\lambda_g - \xi_g(1))\, h^{(i)}_{g,t} + (\lambda_b - \xi_b(-1))\, h^{(i)}_{b,t} + y^{(i)}_{g,t} - y^{(i)}_{b,t}$.

3. For each $i \in 1, ..., N$, compute the normalized importance weight as follows:

$$w^{(i)}_t = \frac{\bar{w}^{(i)}_t}{\sum_{k=1}^{N} \bar{w}^{(k)}_t}.$$

4. For each $i \in 1, ..., N$, the conditional variance process is updated as follows:

$$h^{(i)}_{z,t+1} = w_z + b_z h^{(i)}_{z,t} + \frac{a_z}{h^{(i)}_{z,t}}\left(z^{(i)}_t - c_z h^{(i)}_{z,t}\right)^2 + d_z y^{(i)}_{g,t} + e_z y^{(i)}_{b,t},$$

[3]We set $N = 5000$.

$$h_{g,t+1}^{(i)} = w_g + b_g h_{g,t}^{(i)} + \frac{a_g}{h_{z,t}^{(i)}} \left(z_t^{(i)} - c_g h_{z,t}^{(i)} \right)^2 + d_g y_{g,t}^{(i)} + e_g y_{b,t}^{(i)},$$

$$h_{b,t+1}^{(i)} = w_b + b_b h_{b,t}^{(i)} + \frac{a_b}{h_{z,t}^{(i)}} \left(z_t^{(i)} - c_b h_{z,t}^{(i)} \right)^2 + d_b^{(i)} y_{g,t} + e_b^{(i)} y_{b,t},$$

where $z_t^{(i)} = R_t - m_t^{(i)}$.

5. The filtered variable is calculated using the normalized importance weights obtained from the third step as follows:

$$\begin{cases} \tilde{z}_t = & \sum_{i=1}^{N} z_t^{(i)} w_t^{(i)} \\ \tilde{y}_{g,t} = & \sum_{i=1}^{N} y_{g,t}^{(i)} w_t^{(i)} \\ \tilde{y}_{b,t} = & \sum_{i=1}^{N} y_{b,t}^{(i)} w_t^{(i)} \\ \tilde{h}_{z,t+1} = & \sum_{i=1}^{N} h_{z,t}^{(i)} w_t^{(i)} \\ \tilde{h}_{g,t+1} = & \sum_{i=1}^{N} h_{g,t}^{(i)} w_t^{(i)} \\ \tilde{h}_{b,t+1} = & \sum_{i=1}^{N} h_{b,t}^{(i)} w_t^{(i)} \end{cases}.$$

6. We draw N particles from a set of particle clusters in the smoothed empirical cumulative density function presented in Malik and Pitt (2011) and assume that $\left\{ h_{z,t+1}^{(j_i)} \right\}$, $\left\{ h_{g,t+1}^{(j_i)} \right\}$, and $\left\{ h_{b,t+1}^{(j_i)} \right\}$ are the conditional variances after resampling. The values obtained after resampling are used to replace the original conditional variances:

$$\begin{cases} h_{z,t+1}^{(i)} \leftarrow & h_{z,t+1}^{(j_i)} \\ h_{g,t+1}^{(i)} \leftarrow & h_{g,t+1}^{(j_i)} \\ h_{b,t+1}^{(i)} \leftarrow & h_{b,t+1}^{(j_i)} \end{cases}.$$

Finally, L_{return} can be obtained from the particle filtering algorithm:

$$L_{return}(\Theta) = \sum_{t=1}^{T} \log \left(\sum_{i=1}^{N} \bar{w}_t^{(i)} \right).$$